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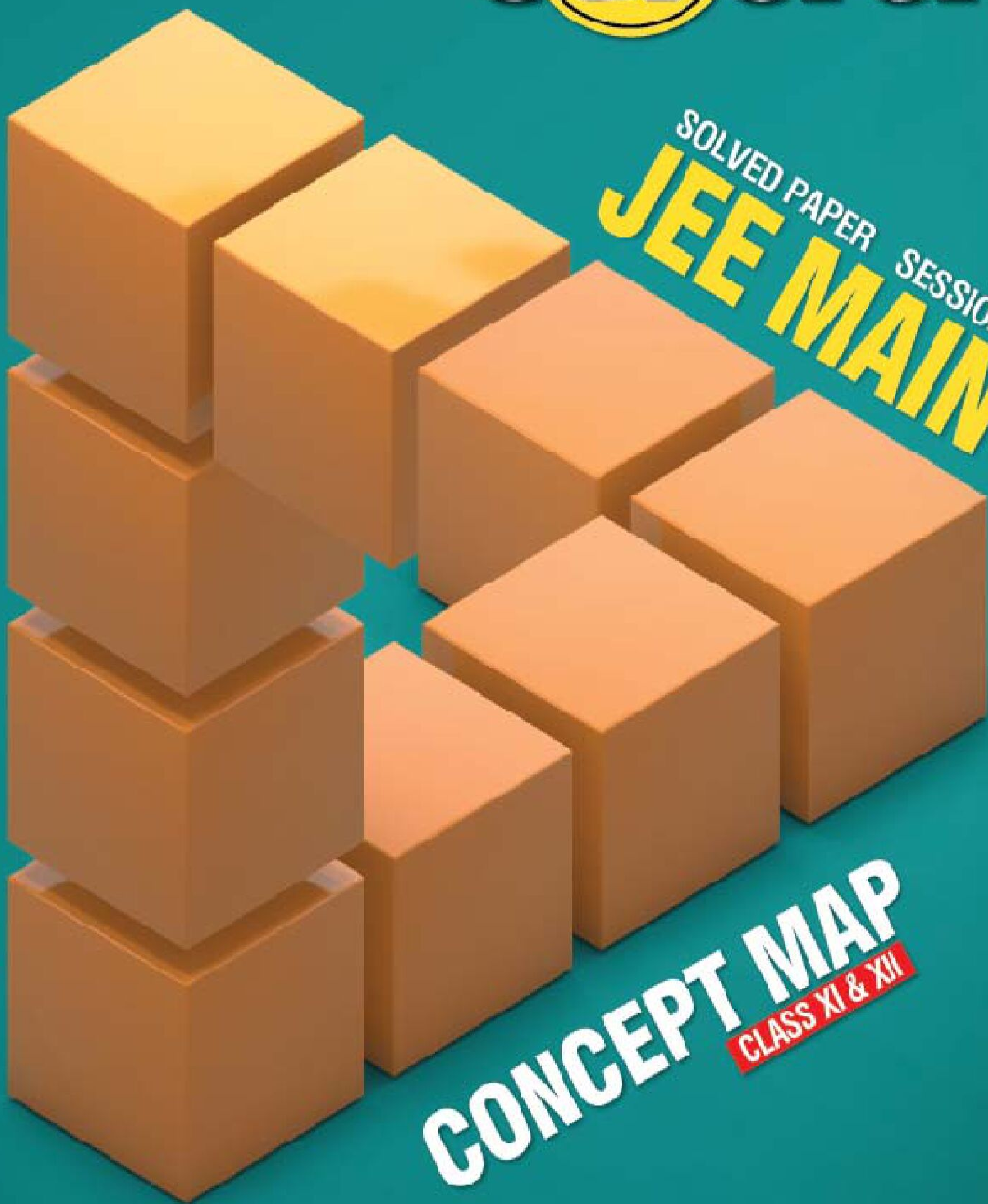

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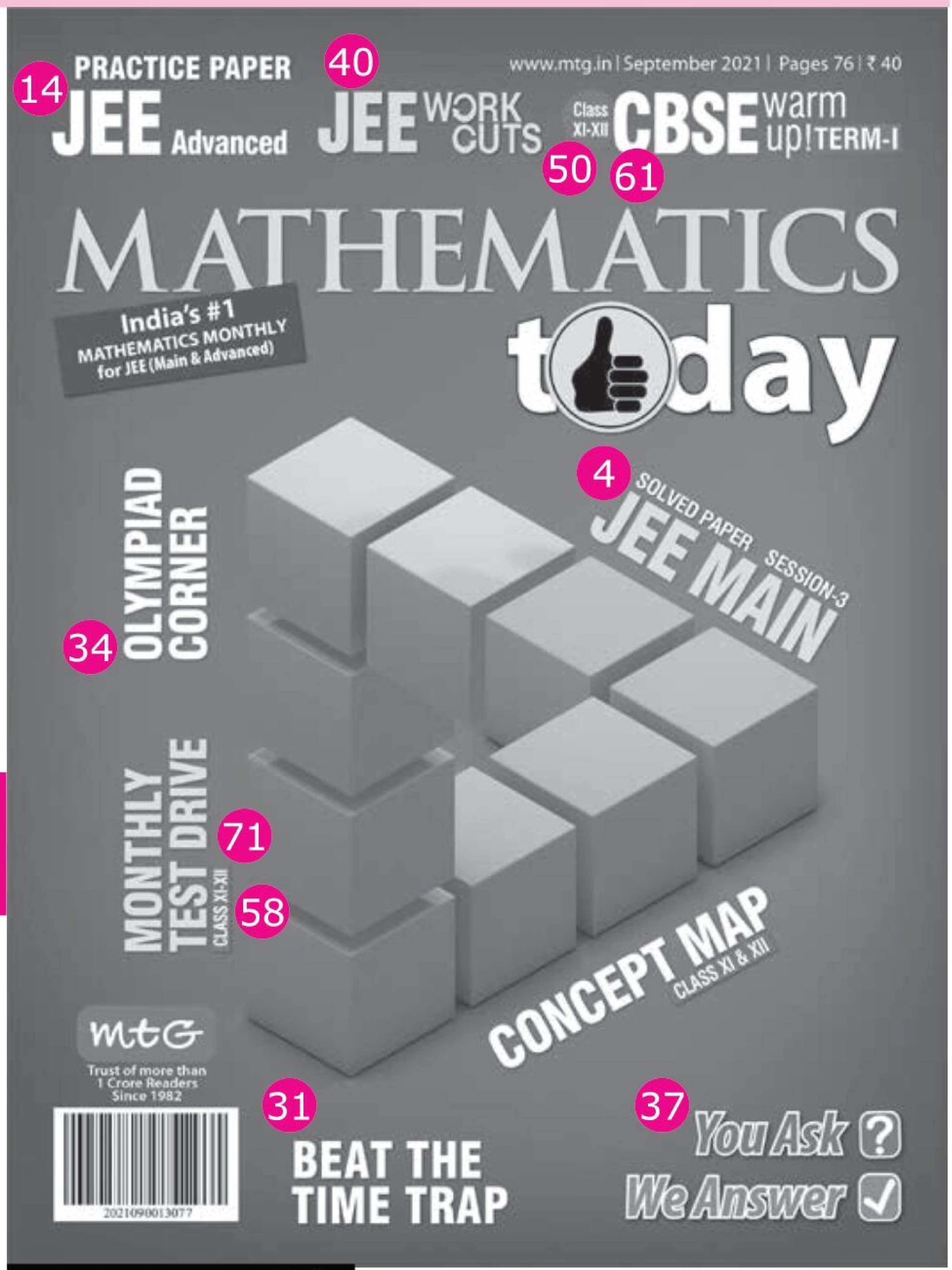

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SOLVED PAPER

Held on 20th July, Morning Shift

JEE MAIN 2021

SECTION-A (MULTIPLE CHOICE QUESTIONS)

1. Let $A = [a_{ij}]$ be a 3×3 matrix, where
$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ -x, & \text{if } |i-j| = 1 \\ 2x+1, & \text{otherwise} \end{cases}$$
Let a function $f: R \to R$ be defined as $f(x) = \det(A)$. Then the sum of maximum and minimum values of f on R is equal to
(a) $\frac{88}{27}$ (b) $\frac{20}{27}$ (c) $-\frac{88}{27}$ (d) $-\frac{20}{27}$

2. If z and ω are two complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{3\pi}{2}$, then $\arg\left(\frac{1-2\bar{z}\omega}{1+2\bar{z}\omega}\right)$ is
(a) $-\frac{\pi}{4}$ (b) $-\frac{3\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{4}$

3. The value of the integral $\int_{-1}^{1} \log_e(\sqrt{1-x} + \sqrt{1+x})dx$ is equal to
(a) $\frac{1}{2}\log_e 2 + \frac{\pi}{4} - \frac{3}{2}$ (b) $2\log_e 2 + \frac{\pi}{4} - 1$
(c) $\log_e 2 + \frac{\pi}{2} - 1$ (d) $2\log_e 2 + \frac{\pi}{2} - \frac{1}{2}$

4. The mean of 6 distinct observations is 6.5 and their variance is 10.25. If 4 out of 6 observations are 2, 4, 5 and 7, then the remaining two observations are
(a) 8, 13 (b) 1, 20 (c) 10, 11 (d) 3, 18

5. Let $y = y(x)$ be the solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \left(\frac{y}{x}\right)dy = 0$, $y(1) = -1$. Then the value of $(y(3))^2$ is equal to
(a) $1 + 4e^6$ (b) $1 - 4e^3$
(c) $1 - 4e^6$ (d) $1 + 4e^3$

6. Let $y = y(x)$ be the solution of the differential equation $x\tan\left(\frac{y}{x}\right)dy = \left(y\tan\left(\frac{y}{x}\right) - x\right)dx$, $-1 \le x \le 1$, $y\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Then the area of the region bounded by the curves $x = 0$, $x = \frac{1}{\sqrt{2}}$ and $y = y(x)$ in the upper half plane is
(a) $\frac{1}{8}(\pi-1)$ (b) $\frac{1}{6}(\pi-1)$
(c) $\frac{1}{12}(\pi-3)$ (d) $\frac{1}{4}(\pi-2)$

7. The number of real roots of the equation $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{4}$ is
(a) 2 (b) 0 (c) 4 (d) 1

8. Let the tangent to the parabola $S: y^2 = 2x$ at the point $P(2, 2)$ meet the x-axis at Q and normal at it meet the parabola S at the point R. Then the area (in sq.units) of the triangle PQR is equal to
(a) 25 (b) $\frac{15}{2}$ (c) $\frac{25}{2}$ (d) $\frac{35}{2}$

9. Let a function $f: R \to R$ be defined as
$$f(x) = \begin{cases} \sin x - e^x & \text{if } x \le 0 \\ a + [-x] & \text{if } 0 < x < 1 \\ 2x - b & \text{if } x \ge 1 \end{cases}$$
where $[x]$ is the greatest integer less than or equal to x. If f is continuous on R, then $(a + b)$ is equal to
(a) 3 (b) 4 (c) 5 (d) 2

10. Let 'a' be a real number such that the function $f(x) = ax^2 + 6x - 15$, $x \in R$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Then the function $g(x) = ax^2 - 6x + 15$, $x \in R$ has a

(a) local maximum at $x = \frac{3}{4}$

(b) local minimum at $x = -\frac{3}{4}$

(c) local maximum at $x = -\frac{3}{4}$

(d) local minimum at $x = \frac{3}{4}$

11. Let $A = \begin{bmatrix} 2 & 3 \\ a & 0 \end{bmatrix}$, $a \in R$ be written as $P + Q$, where P is a symmetric matrix and Q is a skew symmetric matrix. If det(Q) = 9, then the modulus of the sum of all possible values of determinant of P is equal to

(a) 36 (b) 24 (c) 18 (d) 45

12. If in a triangle ABC, AB = 5 units, $\angle B = \cos^{-1}\left(\frac{3}{5}\right)$ and radius of circumcircle of ΔABC is 5 units, then the area (in sq. units) of ΔABC is

(a) $8+2\sqrt{2}$ (b) $10+6\sqrt{2}$

(c) $4+2\sqrt{3}$ (d) $6+8\sqrt{3}$

13. Let a be a positive real number such that $\int_0^a e^{x-[x]}dx = 10e - 9$, where $[x]$ is the greatest integer less than or equal to x. Then a is equal to

(a) $10 + \log_e 2$ (b) $10 + \log_e 3$

(c) $10 + \log_e(1 + e)$ (d) $10 - \log_e(1 + e)$

14. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c} = |\vec{c}|, |\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$ is $\frac{\pi}{6}$, then the value of $|(\vec{a}\times\vec{b})\times\vec{c}|$ is

(a) 3 (b) 4 (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

15. The probability of selecting integers $a \in [-5, 30]$ such that $x^2 + 2(a + 4)x - 5a + 64 > 0$, for all $x \in R$, is

(a) $\frac{1}{4}$ (b) $\frac{2}{9}$

(c) $\frac{7}{36}$ (d) $\frac{1}{6}$

16. Words with or without meaning are to be formed using all the letters of the word EXAMINATION. The probability that the letter M appears at the fourth position in any such word is

(a) $\frac{2}{11}$ (b) $\frac{1}{11}$ (c) $\frac{1}{9}$ (d) $\frac{1}{66}$

17. The Boolean expression $(p \wedge \sim q) \Rightarrow (q \vee \sim p)$ is equivalent to

(a) $\sim q \Rightarrow p$ (b) $p \Rightarrow \sim q$

(c) $p \Rightarrow q$ (d) $q \Rightarrow p$

18. The coefficient of x^{256} in the expansion of $(1 - x)^{101}(x^2 + x + 1)^{100}$ is

(a) ${}^{100}C_{16}$ (b) ${}^{100}C_{15}$ (c) $-{}^{100}C_{16}$ (d) $-{}^{100}C_{15}$

19. Let $[x]$ denote the greatest integer $\le x$, where $x \in R$. If the domain of the real valued function $f(x) = \sqrt{\frac{|[x]|-2}{|[x]|-3}}$ is $(-\infty, a) \cup [b, c) \cup [4, \infty)$, $a < b < c$, then the value of $a + b + c$ is

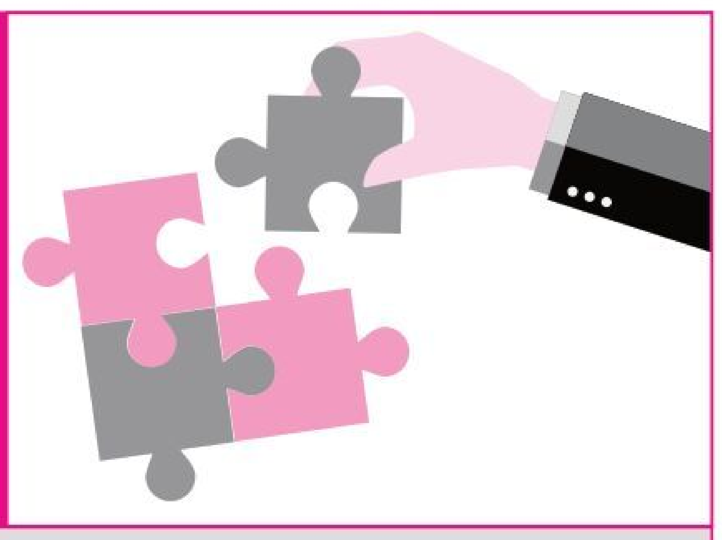
(a) 8 (b) −2 (c) −3 (d) 1

20. If α and β are the distinct roots of the equation $x^2 + (3)^{1/4}x + 3^{1/2} = 0$, then the value of $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$ is equal to

(a) 52×3^{24} (b) 56×3^{25}

(c) 28×3^{25} (d) 56×3^{24}

PUZZLE CORNER



MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics. In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is combined using a specified operation on its numbers. For example, if that value is 3 for a two-box cluster and operation is multiply, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

72×		4−		60×	
	16+				
16+				18×	
	9+	3÷			2−
			36×		
	2−			3−	

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SECTION-B (NUMERICAL VALUE TYPE)

Attempt any 5 questions out of 10.

21. If the shortest distance between the lines $\vec{r}_1 = \alpha\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}), \lambda \in R, \alpha > 0$ and $\vec{r}_2 = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}), \mu \in R$ is 9, then, α is equal to ______.

22. Let a, b, c, d be in arithmetic progression with common difference λ. If $\begin{vmatrix} x+a-c & x+b & x+a \\ x-1 & x+c & x+b \\ x-b+d & x+d & x+c \end{vmatrix} = 2$, then value of λ^2 is equal to ______.

23. Let $y = mx + c, m > 0$ be the focal chord of $y^2 = -64x$, which is tangent to $(x + 10)^2 + y^2 = 4$. Then the value of $4\sqrt{2}(m+c)$ is equal to ______.

24. There are 15 players in a cricket team, out of which 6 are bowlers, 7 are batsmen and 2 are wicket keepers. The number of ways, a team of 11 players be selected from them so as to include at least 4 bowlers, 5 batsmen and 1 wicket keeper, is ________.

25. If the value of $\lim_{x\to 0}(2-\cos x\sqrt{\cos 2x})^{\left(\frac{x+2}{x^2}\right)}$ is equal to e^a, then a is equal to ______.

26. Let $\vec{a}, \vec{b}, \vec{c}$ be three mutally perpendicular vectors of the same magnitude and equally inclined at an angle θ, with the vector $\vec{a} + \vec{b} + \vec{c}$. Then $36\cos^2 2\theta$ is equal to ______.

27. Let P be a plane passing through points (1, 0, 1), (1, –2, 1) and (0, 1, –2). Let a vector $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ be such that $\vec{a}$ is parallel to the plane P, perpendicular to $(\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{a}\cdot(\hat{i} + \hat{j} + 2\hat{k}) = 2$, then $(\alpha - \beta + \gamma)^2$ equals _______.

28. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = 7A^{20} - 20A^7 + 2I$, where I is an identity matrix of order 3×3. If $B = [b_{ij}]$, then b_{13} is equal to _____.

29. Let T be the tangent to the ellipse $E : x^2 + 4y^2 = 5$ at the point $P(1, 1)$. If the area of the region bounded by tangent T, ellipse E, lines $x = 1$ and $x = \sqrt{5}$ is $\alpha\sqrt{5} + \beta + \gamma\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$, then $|\alpha + \beta + \gamma|$ is equal to______.

30. The number of rational terms in the binomial expansion of $\left(4^{\frac{1}{4}} + 5^{\frac{1}{6}}\right)^{120}$ is ______.

SOLUTIONS

1. (c) : We have, $A = \begin{bmatrix} 1 & -x & 2x+1 \\ -x & 1 & -x \\ 2x+1 & -x & 1 \end{bmatrix}$

$\therefore\ f(x) = \det(A) = 4x^3 - 4x^2 - 4x$

$\Rightarrow f'(x) = 12x^2 - 8x - 4 = 4(x - 1)(3x + 1)$

Now, $f'(x) = 0 \Rightarrow x = 1, -\frac{1}{3}$

Also, $f''(x) = 24x - 8$

At $x = 1, f''(x) = 16 > 0$

At $x = -\frac{1}{3}, f''(x) = -16 < 0$

$\therefore$ Sum of maximum and minimum values of f

$= f\left(\frac{-1}{3}\right) + f(1) = \frac{20}{27} - 4 = \frac{-88}{27}$

2. (b) : Let $z = r_1 e^{i\theta}$ and $\omega = r_2 e^{i\phi}$

$\Rightarrow \bar{z} = r_1\, e^{-i\theta}$

Given $|z\omega| = 1 \Rightarrow |r_1 r_2\, e^{i(\theta+\phi)}| = 1$

$\Rightarrow r_1 r_2 = 1$ ...(i)

Given $\arg(z) - \arg(\omega) = \frac{3\pi}{2}$

$\Rightarrow \theta - \phi = \frac{3\pi}{2}$ ...(ii)

Now, $\bar{z}\omega = r_1 e^{-i\theta} \,.\, r_2 e^{i\phi}$

$= r_1 r_2\, e^{-i(\theta-\phi)}$

$= \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}$ [From (i) and (ii)]

$\Rightarrow \bar{z}\omega = i$

$\therefore\ \arg\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right) = \arg\left(\frac{1-2i}{1+3i}\right)$

$= \arg\left(\frac{1-2i}{1-3i} \times \frac{1-3i}{1-3i}\right) = \arg\left(\frac{-1}{2}(1+i)\right)$

$= -\pi + \tan^{-1}(1)$

$[\because \arg(z) = -\pi + \tan^{-1}\frac{y}{x}$, if $x < 0$ and $y < 0]$

$= -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$

3. (c) : Let $I = 2\int_0^1 \log_e(\sqrt{1-x} + \sqrt{1+x}) \cdot 1\,dx$

$(\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)dx$, if $f(x) = f(-x))$

$= 2\Big[\left(\log_e(\sqrt{1-x} + \sqrt{1+x}) \cdot x\right)_0^1$

$-\int_0^1 x\left(\frac{1}{\sqrt{1-x} + \sqrt{1+x}}\right) \cdot \left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}\right)dx\Big]$

$$=2\left[\left(\log_e \sqrt{2}-0\right)\right] - \int_0^1 \frac{x(\sqrt{1-x}-\sqrt{1+x})}{(\sqrt{1-x}+\sqrt{1+x})\sqrt{1-x^2}}dx$$

$$=\log_e 2+\int_0^1\left(\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)dx$$

$$=\log_e 2+(\sin^{-1}x-x)_0^1 \;=\log_e 2+\frac{\pi}{2}-1$$

4. (c) : Let the remaining two observations be a and b.

Then, mean = $\frac{2+4+5+7+a+b}{6}$ = 6.5 (Given)

$\Rightarrow\; a+b=21$ ...(i)

Also, variance = 10.25 (Given)

$\Rightarrow\; \frac{1}{6}(2^2+4^2+5^2+7^2+a^2+b^2)-(6.5)^2=10.25$

$\Rightarrow\; a^2+b^2=315-94=221$ ...(ii)

Now, $(a+b)^2+(a-b)^2=2(a^2+b^2)$

$\Rightarrow\; (a-b)^2=2\times 221-(21)^2$ [From (i) and (ii)]

$\Rightarrow\; (a-b)^2=1 \Rightarrow a-b=\pm 1$

If $a-b=1$, then $a=11$, $b=10$

If $a-b=-1$, then $a=10$, $b=11$

5. (c) : We have, $e^x\sqrt{1-y^2}dx+\left(\frac{y}{x}\right)dy=0$

$\Rightarrow\; x\,e^x\,dx=\frac{-y}{\sqrt{1-y^2}}dy$

Integrating both sides, we get

$$\int xe^x dx=-\int\frac{2y}{2\sqrt{1-y^2}}dy$$

$\Rightarrow\; xe^x-\int e^x\,dx=\sqrt{1-y^2}+c$

$\Rightarrow\; xe^x-e^x=\sqrt{1-y^2}+c$

At $y(1)=-1$, $c=0$

$\therefore\; e^x(x-1)=\sqrt{1-y^2}$

$\Rightarrow\; y^2=1-(e^x(x-1))^2$

$\therefore\; (y(3))^2=1-4e^6$

6. (a) : We gave,

$$x\tan\left(\frac{y}{x}\right)dy=\left(y\tan\left(\frac{y}{x}\right)-x\right)dx$$

$$\Rightarrow\; \frac{dy}{dx}=\frac{x\left(\frac{y}{x}\tan\left(\frac{y}{x}\right)-1\right)}{x\tan\left(\frac{y}{x}\right)}$$

$$\Rightarrow\; \frac{dy}{dx}=\frac{y}{x}-\cot\left(\frac{y}{x}\right) \quad \text{...(i)}$$

Putting $\frac{y}{x}=v \Rightarrow y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

From (i), $v+x\frac{dv}{dx}=v-\cot(v) \Rightarrow x\frac{dv}{dx}=\frac{-1}{\tan v}$

$$\Rightarrow\; \int(\tan v)dv=-\int\frac{1}{x}dx$$

$$\Rightarrow\; \log\left|\sec\left(\frac{y}{x}\right)\right|=-\log|x|+c \quad \text{...(iii)}$$

At $y\left(\frac{1}{2}\right)=\frac{\pi}{6}$, $c=0$

$\therefore\; \sec\left(\frac{y}{x}\right)=\frac{1}{x} \Rightarrow \cos\left(\frac{y}{x}\right)=x \Rightarrow y=x\cos^{-1}x$

$\therefore$ Required area

$$=\int_0^{1/\sqrt{2}} y\,dx=\int_0^{1/\sqrt{2}}(x\cos^{-1}x)dx$$

$$=\left[(2x^2-1)\frac{\cos^{-1}x}{4}-\frac{x}{4}\sqrt{1-x^2}\right]_0^{1/\sqrt{2}}=-\frac{1}{8}+\frac{\pi}{8}$$

$$=\frac{1}{8}(\pi-1)$$

7. (b): We have,

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{4} \quad \text{...(i)}$$

The equation (i) is defined for $x^2+x\geq 0$

$\Rightarrow\; x(x+1)\geq 0$

$\Rightarrow\; x\geq 0$ & $x\leq -1$ and $0\leq x^2+x+1\leq 1$

$\Rightarrow\; -1\leq x\leq 0$

$\Rightarrow\; x=0,-1$ is the only solution but it does not satisfy (i).

8. (c) : Equation of tangent to the parabola $y^2=2x$ at the point $P(2, 2)$ is given by $yy_1=2a(x+x_1)$

$\therefore\; y(2)=2\times\left(\frac{1}{2}\right)(x+2)$

$\Rightarrow\; 2y=x+2$ ...(i)

Equation (i) meet the x-axis at Q $\therefore$ $Q=(-2, 0)$

Equation of normal to the parabola $y^2=2x$ at point $P(2, 2)$ is given by

$$y-2=-\frac{2}{2\left(\frac{1}{2}\right)}(x-2) \Rightarrow y-2=-2(x-2)$$

$\Rightarrow\; y=6-2x$ ...(ii)

Equation (ii) meet the $y^2 = 2x$ at R $\therefore$ $R\left(\frac{9}{2}, -3\right)$

$$\therefore \quad \text{Area }(\Delta PQR) = \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ -2 & 0 & 1 \\ \frac{9}{2} & -3 & 1 \end{vmatrix}$$

$$= \frac{25}{2} \text{ sq. units}$$

9. (a) : We have, $f(x)$ is Continuous at $x = 0$.

$\therefore \quad \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

$\Rightarrow \quad a - 1 = 0 - e^0$

$\Rightarrow \quad a - 1 = -1 \Rightarrow a = 0$

Also, $f(x)$ is continuous at $x = 1$.

$\therefore \quad \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$

$\Rightarrow \quad 2(1) - b = a + (-1)$

$\Rightarrow \quad b = 2 - a + 1$

$\Rightarrow \quad a + b = 3$

10. (c) : Since, $f(x) = ax^2 + 6x - 15$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$.

$\therefore \quad f(x)$ has a local maxima at $x = \frac{3}{4}$.

Now, $f'(x) = 2ax + 6$

$\Rightarrow \quad f'\left(\frac{3}{4}\right) = 0$

$\Rightarrow \quad 2a\left(\frac{3}{4}\right) + 6 = 0 \Rightarrow a = -4$

$\therefore \quad g(x) = -4x^2 - 6x + 15 \Rightarrow g'(x) = -8x - 6$

Now, $g'(x) = 0$

$\Rightarrow \quad -8x - 6 = 0 \Rightarrow x = -\frac{3}{4}$

$g''(x) = -8 < 0$, at $x = -\frac{3}{4}$.

$\therefore \quad g(x)$ has local maximum at $x = -\frac{3}{4}$.

11. (a) : We have, $A = \begin{bmatrix} 2 & 3 \\ a & 0 \end{bmatrix}$ $\therefore$ $A^T = \begin{bmatrix} 2 & a \\ 3 & 0 \end{bmatrix}$

We can write A as, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q$

$$\therefore \quad P = \begin{bmatrix} 2 & \frac{3+a}{2} \\ \frac{3+a}{2} & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & \frac{3-a}{2} \\ \frac{-(3-a)}{2} & 0 \end{bmatrix}$$

Now, det $(Q) = 9 \Rightarrow 0 + \left(\frac{3-a}{2}\right)^2 = 9$

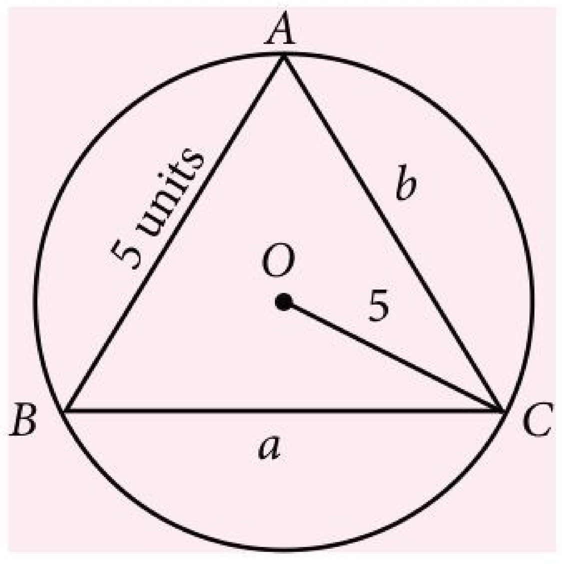
$\Rightarrow (3 - a) = \pm 6 \Rightarrow \quad a = -3, 9$

At $a = -3$, $P = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, det $(P) = 0$

At $a = 9$, $P = \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix}$, det $(P) = -36$

$\therefore \quad$ Required sum $= |0 + (-36)| = |-36| = 36$

12. (d): Given, $\cos B = \frac{3}{5}$



$\therefore \quad \sin B = \frac{4}{5}$

We have, $R = 5$

$\because \quad \frac{c}{\sin C} = 2R$

SAMURAI SUDOKU

ANSWER - AUGUST 2021

2	6	1	3	4	9	8	7	5				7	2	6	4	3	8	5	9	1
4	3	7	8	1	5	2	6	9				9	3	1	5	2	7	6	4	8
8	5	9	2	7	6	3	1	4				8	5	4	6	9	1	7	2	3
6	4	2	7	8	3	9	5	1				3	9	7	8	5	2	4	1	6
3	1	5	6	9	2	7	4	8				1	4	8	9	7	6	3	5	2
7	9	8	4	5	1	6	3	2				2	6	5	3	1	4	9	8	7
5	2	3	1	6	8	4	9	7	3	5	8	6	1	2	7	4	5	8	3	9
1	8	4	9	3	7	5	2	6	9	7	1	4	8	3	2	6	9	1	7	5
9	7	6	5	2	4	1	8	3	4	2	6	5	7	9	1	8	3	2	6	4
						8	3	4	6	9	2	1	5	7						
						6	7	9	1	3	5	8	2	4						
						2	5	1	8	4	7	3	9	6						
4	5	6	7	8	3	9	1	2	5	6	4	7	3	8	9	4	2	5	6	1
7	8	9	4	2	1	3	6	5	7	8	9	2	4	1	5	3	6	8	7	9
3	2	1	6	9	5	7	4	8	2	1	3	9	6	5	7	1	8	4	2	3
6	4	5	2	1	7	8	9	3				8	1	3	2	6	5	9	4	7
1	9	3	5	6	8	2	7	4				6	7	4	1	8	9	2	3	5
8	7	2	9	3	4	1	5	6				5	2	9	4	7	3	1	8	6
9	6	7	8	5	2	4	3	1				1	9	7	3	2	4	6	5	8
5	1	8	3	4	9	6	2	7				3	8	2	6	5	1	7	9	4
2	3	4	1	7	6	5	8	9				4	5	6	8	9	7	3	1	2

$\therefore \quad \sin C = \frac{5}{10} \Rightarrow C = 30°$

Now, $\frac{b}{\sin B} = 2R \Rightarrow b = 2 \times 5 \times \frac{4}{5} = 8$

Now, by cosine formula, we have

$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow \frac{3}{5} = \frac{a^2 + 25 - 64}{2 \times a \times 5}$

$\Rightarrow \quad a^2 - 6a - 3 = 0$

$\Rightarrow \quad a = \frac{6 \pm \sqrt{192}}{2} \Rightarrow a = 3 + 4\sqrt{3} \qquad (\because a \neq 3 - 4\sqrt{3})$

Now, Area $(\Delta ABC) = \frac{abc}{4R} = \frac{(3+4\sqrt{3})(8)(5)}{(4)(5)} = 2(3+4\sqrt{3})$

$= (6 + 8\sqrt{3})$ sq. units

13. (a) : a is positive integer *i.e.* $a > 0 \therefore a = [a] + \{a\}$, where $[\cdot]$ and $\{\cdot\}$ denote the G.I.F. and fractional part

Let $n \le a < n + 1, n \in W$

Here, $[a] = n$

Now $\int_0^a e^{x-[x]} dx = 10e - 9$

$\Rightarrow \quad \int_0^n e^{\{x\}} dx + \int_n^a e^{x-[x]} dx = 10e - 9$

$\therefore \quad n\int_0^1 e^x dx + \int_n^a e^{x-n} dx = 10e - 9$

$\Rightarrow \quad n(e - 1) + (e^{a-n} - 1) = 10e - 9$

$\therefore \quad n = 10$ and $\{a\} = \log_e 2$

So, $a = [a] + \{a\} = 10 + \log_e 2$

14. (c) : We have, $|\vec{a}| = 3$ and $\vec{a} \cdot \vec{c} = |\vec{c}|$

Now, $|\vec{c} - \vec{a}| = 2\sqrt{2}$

$\Rightarrow \quad |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c} \cdot \vec{a} = 8$

$\Rightarrow \quad |\vec{c}|^2 + 9 - 2|\vec{c}| = 8$

$\Rightarrow \quad |\vec{c}|^2 - 2|\vec{c}| + 1 = 0 \Rightarrow |\vec{c} - 1|^2 = 0 \quad \Rightarrow \quad |\vec{c}| = 1$

Also, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = 3$

$\therefore \quad |(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}| \sin\frac{\pi}{6} = 3 \times 1 \times \frac{1}{2} = \frac{3}{2}$

15. (b): We have, $x^2 + 2(a + 4)x - 5a + 64 > 0 \ \forall \ x \in R$

$\therefore \quad D < 0$

$\Rightarrow \quad 4(a + 4)^2 - 4(-5a + 64) \times 1 < 0$

$\Rightarrow \quad a^2 + 16 + 8a + 5a - 64 < 0$

$\Rightarrow \quad a^2 + 13a - 48 < 0 \Rightarrow (a + 16)(a - 3) < 0$

$\Rightarrow \quad a \in (-16, 3)$

$\therefore$ Possibilities for a are $\{-5, -4, ..., 2\}$ *i.e.*, 8 in number.

In set $[-5, 30]$, total integers = 36

$\therefore$ Required probability $= \frac{8}{36} = \frac{2}{9}$

16. (b): In the word EXAMINATION, "A, N and I" appears two times.

$\therefore$ Total number of words with M at fourth place

$= \frac{10!}{2! \cdot 2! \cdot 2!}$

Also, total number of words formed using all letters of word EXAMINATION $= \frac{11!}{2! \cdot 2! \cdot 2!}$

$\therefore$ Required probability $= \frac{10!}{11!} = \frac{1}{11}$

17. (c) :

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$q \vee \sim p$	$(p \wedge \sim q) \Rightarrow (q \vee \sim p)$	$p \Rightarrow q$
T	F	F	T	T	F	F	F
T	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	T	T	F	F	T	T	T

18. (b): We have, $(1 - x)^{101} \cdot (x^2 + x + 1)^{100}$

$= (1 - x)^{100} \cdot (1 - x)(x^2 + x + 1)^{100}$

$= ((1 - x)(x^2 + x + 1))^{100} \cdot (1 - x)$

$= (1 - x^3)^{100}(1 - x)$

$= (1 - x^3)^{100} - x(1 - x^3)^{100}$

$\therefore$ Required coefficient $= (-1) \times (-{}^{100}C_{85})$

$= {}^{100}C_{85} = {}^{100}C_{15}$

19. (b): For domain, $\frac{|[x]| - 2}{|[x]| - 3} \ge 0$

Case I : When $|[x]| - 2 \ge 0$ and $|[x]| - 3 > 0$

$\therefore \quad x \in (-\infty, -3) \cup [4, \infty)$ ...(i)

Case II : When $|[x]| - 2 \le 0$ and $|[x]| - 3 < 0$

$\therefore \quad x \in [-2, 3)$ ...(ii)

From (i) and (ii), we get

Domain of function $= (-\infty, -3) \cup [-2, 3) \cup [4, \infty)$

$\therefore \quad (a + b + c) = -3 + (-2) + 3 = -2$

20. (a) : As, $(\alpha^2 + \sqrt{3}) = -(3)^{1/4} \alpha$

$\Rightarrow \quad (\alpha^4 + 2\sqrt{3}\alpha^2 + 3) = \sqrt{3}\alpha^2$

(On squaring both sides)

$\therefore \quad (\alpha^4 + 3) = -\sqrt{3}\alpha^2$

$\Rightarrow \quad \alpha^8 + 6\alpha^4 + 9 = 3\alpha^4$ (Again squaring both sides)

$\Rightarrow \quad \alpha^8 = -9 - 3\alpha^4$

Multiplying by α^4, we get

$\alpha^{12} = -9\alpha^4 - 3\alpha^8$

$\Rightarrow \alpha^{12} = -9\alpha^4 - 3(-9 - 3\alpha^4)$

$\Rightarrow \alpha^{12} = -9\alpha^4 + 27 + 9\alpha^4 \Rightarrow \alpha^{12} = 27$

$\Rightarrow (\alpha^{12})^8 = (27)^8$

$\Rightarrow \alpha^{96} = (3)^{24}$

Similarly, $\beta^{96} = (3)^{24}$

$\therefore \alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1) = 2 \times 26 \times 3^{24}$

$= (3)^{24} \times 52$

21. (6): We know that, if $\vec{r}_1 = \vec{a} + \lambda\vec{b}$ and $\vec{r}_2 = \vec{c} + \lambda\vec{d}$, then shortest distance between two lines is given by

$$d = \frac{|(\vec{b}\times\vec{d})\cdot(\vec{a}-\vec{c})|}{|\vec{b}\times\vec{d}|}$$

Now, $\vec{a}-\vec{c} = (\alpha+4)\hat{i} + 2\hat{j} + 3\hat{k}$

Also, $\vec{b}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = 8\hat{i} + 8\hat{j} + 4\hat{k}$

$\therefore |\vec{b}\times\vec{d}| = 12$

Now, $\dfrac{\vec{b}\times\vec{d}}{|\vec{b}\times\vec{d}|} = \dfrac{2\hat{i}+2\hat{j}+\hat{k}}{3}$

$\therefore$ Required distance

$$= \frac{((\alpha+4)\hat{i}+2\hat{j}+3\hat{k})(2\hat{i}+2\hat{j}+\hat{k})}{3} = 9$$

$\Rightarrow 2\alpha + 8 + 4 + 3 = 27 \Rightarrow 2\alpha = 27 - 15$

$\Rightarrow 2\alpha = 12 \Rightarrow \alpha = 6$

22. (1): We have, $\begin{vmatrix} x+a-c & x+b & x+a \\ x-1 & x+c & x+b \\ x-b+d & x+d & x+c \end{vmatrix}$

Applying $c_2 \to c_2 - c_3$, we get

$$\begin{vmatrix} x-2\lambda & \lambda & x+a \\ x-1 & \lambda & x+b \\ x+2\lambda & \lambda & x+c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} x-2\lambda & \lambda & x+a \\ 2\lambda-1 & 0 & \lambda \\ 4\lambda & 0 & 2\lambda \end{vmatrix}$$

$= -\lambda(4\lambda^2 - 2\lambda - 4\lambda^2) = 2\lambda^2 = 2$ (Given)

$\therefore \lambda^2 = 1$

23. (34) :We have, $y^2 = -64x$, focus $(-16, 0)$.

Now, $y = mx + c$ is focal chord.

$\Rightarrow c = 16\,m$ ...(i)

Also, $y = mx + c$ is tangent to $(x + 10)^2 + y^2 = 4$

$\therefore y = m(x + 10) \pm 2\sqrt{1+m^2}$

$\Rightarrow c = 10m \pm 2\sqrt{1+m^2}$ (ii)

From (i) and (ii), we get

$16m = 10m \pm 2\sqrt{1+m^2}$

$\Rightarrow 6m = 2\sqrt{1+m^2}$ $(\because m > 0)$

$\Rightarrow 9m^2 = 1 + m^2$

$\Rightarrow m = \dfrac{1}{2\sqrt{2}}$ and $c = \dfrac{8}{\sqrt{2}}$

$\therefore 4\sqrt{2}(m+c) = 4\sqrt{2}\left(\dfrac{17}{2\sqrt{2}}\right) = 34$

24. (777) : Total number of players = 15

Number of bowler = 6

Number of batsman = 7

Number of wicket keepers = 2

$\therefore$ Total number of ways for with at least 4 bowlers, 5 batsman and 1 wicket keeper is to be selected

$= {}^6C_4\,{}^7C_5\,{}^2C_2 + {}^6C_4\,{}^7C_6\,{}^2C_1 + {}^6C_5\,{}^7C_5\,{}^2C_1$

$= 315 + 210 + 252 = 777$

25. (3) : We have, $\lim\limits_{x\to 0}\left(2 - \cos x\sqrt{\cos 2x}\right)^{\frac{x+2}{x^2}}$

$= e^{\lim\limits_{x\to 0}(1-\cos x\sqrt{\cos 2x})\times\frac{x+2}{x^2}}$ (Indeterminate form)

$= e^{\lim\limits_{x\to 0}\frac{(1-\cos x\sqrt{\cos 2x})(x+2)}{x^2}}$

$= e^{\lim\limits_{x\to 0}\frac{(1-\cos x\sqrt{\cos 2x})(1+\cos x\sqrt{\cos 2x})}{(1+\cos x\sqrt{\cos 2x})}\times\frac{x+2}{x^2}}$

$= e^{\lim\limits_{x\to 0}\frac{1-\cos^2 x(\cos 2x)}{1+\cos x\sqrt{\cos 2x}}\times\frac{x+2}{x^2}}$

$= e^{\lim\limits_{x\to 0}\frac{1-\cos^2 x(1-2\sin^2 x)}{1+\cos x\sqrt{\cos 2x}}\times\frac{x+2}{x^2}}$

$= e^{\lim\limits_{x\to 0}\frac{1-(1-\sin^2 x)(1-2\sin^2 x)}{x^2}\cdot\frac{x+2}{1+\cos x\sqrt{\cos 2x}}}$

$= e^{\lim\limits_{x\to 0}\frac{3\sin^2 x-2\sin^4 x}{x^2}\cdot\frac{x+2}{1+\cos x\sqrt{\cos 2x}}}$

$= e^{\lim\limits_{x\to 0}\frac{3\sin^2 x}{x^2}\left(1-\frac{2}{3}\sin^2 x\right)\cdot\frac{x+2}{1+\cos x\sqrt{\cos 2x}}}$

$= e^3 = e^a$ (Given) $\Rightarrow a = 3$

26. (4) : $|\vec{a}+\vec{b}+\vec{c}|^2$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}| = 3$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}$

Now, $\vec{a}(\vec{a}+\vec{b}+\vec{c}) = |a||\vec{a}+\vec{b}+\vec{c}|\cos\theta = 1.\sqrt{3}\cos\theta$

$\Rightarrow 1 = \sqrt{3}\cos\theta$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{3}} \quad \therefore \cos^2\theta = \dfrac{1}{3}$

$\therefore \quad 36\cos^2 2\theta = 36(2\cos^2\theta - 1)^2 = 36\left(\dfrac{2}{3}-1\right)^2 = 4$

27. (81) : Equation of plane passing through (1, 0, 1), (1, –2, 1) and (0, 1, –2) is given by

$$\begin{vmatrix} x-1 & y-0 & z-1 \\ 1-1 & -2-0 & 1-1 \\ 0-1 & 1-0 & -2-1 \end{vmatrix} = 0 \Rightarrow 3x - z - 2 = 0$$

$\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ is parallel to $3x - z - 2 = 0$

$\Rightarrow \quad 3\alpha - \gamma = 0$ ...(i)

$\vec{a}$ is perpendicular to $\hat{i} + 2\hat{j} + 3\hat{k}$

$\Rightarrow \quad \alpha + 2\beta + 3\gamma = 0$ ...(ii)

Also, $\vec{a}\cdot(\hat{i} + \hat{j} + 2\hat{k}) = 2$

$\Rightarrow \quad \alpha + \beta + 2\gamma = 2$ ...(iii)

Solving (i), (ii) and (iii), we get

$\alpha = 1, \beta = -5, \gamma = 3$

$\therefore \quad (\alpha - \beta + \gamma)^2 = 9^2 = 81$

28. (910) : Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = I + P,$

where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now, $B = 7A^{20} - 20A^7 + 2I$

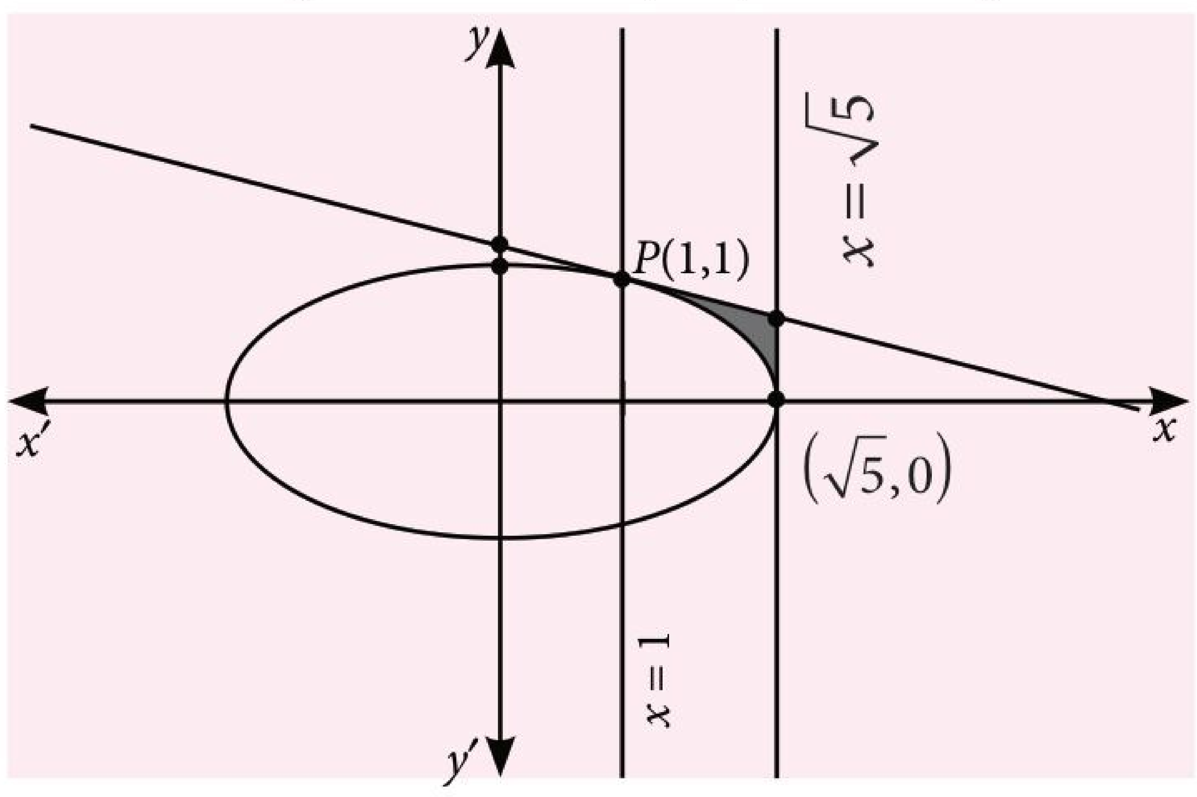
$= 7(I + P)^{20} - 20(I + P)^7 + 2I$

$= 7(I + 20P + {}^{20}C_2P^2) - 20(I + 7P + {}^7C_2P^2) + 2I$

$\therefore \quad b_{13} = 7 \times {}^{20}C_2 - 20 \times {}^7C_2 = 910$

29. (1.25) : We have, $E : x^2 + 4y^2 = 5$

Equation of tangent to E at $P(1,1)$ is : $x + 4y = 5$



Required area

$$= \int_1^{\sqrt{5}} \left(\frac{5-x}{4} - \frac{\sqrt{5-x^2}}{2} \right) dx$$

$$= \left[\frac{5x}{4} - \frac{x^2}{8} - \frac{x}{4}\sqrt{5-x^2} - \frac{5}{4}\sin^{-1}\frac{x}{\sqrt{5}} \right]_1^{\sqrt{5}}$$

$$= \frac{5}{4}\sqrt{5} - \frac{5}{4} - \frac{5}{4}\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

On comparing with given area, we get

$\alpha = \dfrac{5}{4}, \beta = -\dfrac{5}{4}$ and $\gamma = -\dfrac{5}{4}$

$\therefore \quad |\alpha + \beta + \lambda| = \left| \dfrac{5}{4} - \dfrac{5}{4} - \dfrac{5}{4} \right| = 1.25$

30. (21) : General term in the binomial expansion of

$\left(4^{\frac{1}{4}} + 5^{\frac{1}{6}}\right)^{120} = T_{r+1} = {}^{120}C_r(2^{1/2})^{120-r}(5)^{r/6}$

For rational terms, $r = 6k$, $r = 120 - 2k_1$ and $0 \le r \le 120$

$\Rightarrow \quad r = 6k$ *i.e.*, $r = 0, 6, 12, \ldots\ldots, 120$

So, total number of terms are 21.

JEE ADVANCED PRACTICE PAPER

Exam on 3rd October

PAPER-1

SINGLE OPTION CORRECT TYPE

1. Let S be the set of all complex numbers z satisfying $|z-2+i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

2. If $(1+k)\tan^2 x - 4\tan x - 1 + k = 0$ has real roots, then which one of the following is not true?

(a) $k^2 \leq 5$ (b) $\tan(x_1 + x_2) = 2$

(c) for $k = 2, x_1 = \frac{\pi}{4}$ (d) none of these

3. If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, then $\int_1^2 f(x)\,dx =$

(a) $\frac{1}{(a^2+b^2)}\left[a\log 2 - 5a + \frac{7}{2}b\right]$

(b) $\frac{1}{(a^2-b^2)}\left[a\log 2 - 5a + \frac{7}{2}b\right]$

(c) $\frac{1}{(a^2-b^2)}\left[a\log 2 - 5a - \frac{7}{2}b\right]$

(d) $\frac{1}{(a^2+b^2)}\left[a\log 2 - 5a - \frac{7}{2}b\right]$

4. If $A(z_1)$, $B(z_2)$ and $C(z_3)$ are three points in the argand plane, where $|z_1 + z_2| = ||z_1| - |z_2||$ and $|(1-i)z_1 + iz_3| = |z_1| + |z_3 - z_1|$, where $i = \sqrt{-1}$, then

(a) A, B and C lie on a fixed circle with centre $\frac{(z_2+z_3)}{2}$.

(b) A, B and C are collinear points.

(c) ABC form an equilateral triangle.

(d) ABC form an obtuse angle triangle.

5. Given that $x_1 + x_2 + x_3 = 0$, $y_1 + y_2 + y_3 = 0$ and $x_1y_1 + x_2y_2 + x_3y_3 = 0$. Then

$$\frac{x_1^2}{x_1^2+x_2^2+x_3^2} + \frac{y_1^2}{y_1^2+y_2^2+y_3^2} =$$

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{4}{3}$

6. In ΔABC, $\sqrt{3}\sin C = 2\sec A - \tan A$ and one of the sides is of length 2, then maximum area of the triangle ABC is

(a) $2\sqrt{3}$ sq. units (b) $4\sqrt{3}$ sq. units

(c) $\frac{\sqrt{3}}{2}$ sq. units (d) $\frac{\sqrt{3}}{4}$ sq. units

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

7. Let $f: R \to R$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \leq x < 1; \\ (2/3)x^3 - 4x^2 + 7x - (8/3), & 1 \leq x < 3; \\ (x-2)\log_e(x-2) - x + (10/3), & x \geq 3. \end{cases}$$

Then which of the following options is/are correct?

(a) f' is not differentiable at $x = 1$

(b) f is increasing on $(-\infty, 0)$

(c) f is onto

(d) f' has a local maximum at $x = 1$

8. The roots of the equation, $(x^2+1)^2 = x(4x^2+5x+4)$ are both real and imaginary of the form $A \pm \sqrt{B}$ and $C \pm iD$ respectively, then

(a) $D = \frac{\sqrt{3}}{2}$ (b) $A = \frac{5}{2}$

(c) $B = \frac{21}{4}$ (d) $C = -\frac{1}{2}$

9. Bag A contains 2 white and 3 red balls and bag B contains 4 white and 7 red balls, one bag is selected randomly and a ball is drawn from the bag then

(a) If drawn ball is found to be red, the probability that it was drawn from bag B is $\frac{35}{68}$

(b) If drawn ball is found to be white, the probability that it was drawn from bag A is $\frac{11}{21}$

(c) If drawn ball is found to be red, the probability that it was drawn from bag A is $\frac{33}{68}$

(d) If drawn ball is found to be white, the probability that it was drawn from bag B is $\frac{10}{21}$

10. The possible value of the expression

$\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}$ $\left(\alpha, \beta \neq \frac{n\pi}{2},\ n \in I\right)$ is/are

(a) 6 (b) 8 (c) 10 (d) 12

11. Let f be a non negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt,\ 0 \le x \le 1$ and $f(0) = 0$, then

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(d) $f\left(\frac{1}{2}\right) > 0$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

12. The point P is the intersection of the straight line joining the points $Q(2, 3, 5)$ and $R(1, -1, 4)$ with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point $T(2, 1, 4)$ to QR, then the possible length of the line segment PS is/are

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 2 (d) $2\sqrt{2}$

NUMERICAL VALUE TYPE

13. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is ____________.

14. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is ____________.

15. $\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to ____________.

16. If for some natural number n, $(1 + 2 + 3 + ... + n) + k = 2013$, where k is one of the numbers 1, 2, 3, ..., n then $n + k =$ ____________.

17. In a certain test there are n questions. In this test 2^{n-i} students gave wrong answers to atleast i questions, where $i = 1, 2, 3, \ldots, n$. If the total number of wrong answers given is 2047, then n is ____________.

18. Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is meet by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$, then the x-coordinate of the point of intersection of these tangents is ____________.

PAPER-2

SINGLE DIGIT INTEGER ANSWER TYPE

1. If the papers of 5 students can be checked by any one of the 5 teachers. If the probability that all the 5 papers are checked by exactly 2 teachers is m then the value of $\frac{125m}{2}$ equal to ____________.

2. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is 2^{k+4}, where k is equal to ____________.

3. If $(3 + x^{2008} + x^{2009})^{2010} = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, then the value of $a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 + a_6 - ...$ $= K^{1005}$, where K is equal to ____________.

4. Let $u(x)$ and $v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to ______. (u' is equal to derivative of 'u')

5. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}\cdot\vec{b} = 0$. For some $x, y \in R$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a}\times\vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at some angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ____ .

6. If $f(x + y) = f(x) f(y)$ and $f(x) = 1 + xg(x)H(x)$ where $\lim_{x\to 0} g(x) = 2$, $\lim_{x\to 0} H(x) = 3$, then $f'(x) = Kf(x)$ where $K =$ ______.

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

7. Let $f(x) = \frac{x}{1+x^2}$ and $g(x) = \frac{e^{-x}}{1+[x]}$, where $[\cdot]$ is the greatest integer less than or equal to x, then

(a) Domain $(f + g) = R - [-2, 0)$

(b) Domain $(f - g) = R - [-1, 0)$

(c) Range $f \cap$ Range $g = \left[-2, \frac{1}{2}\right]$

(d) Range $g = R - \{0\}$

8. Consider the function $f(x) = \sin^5 x + \cos^5 x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$. Which of the following is/are correct?

(a) $f(x)$ is monotonic increasing in $\left(0, \frac{\pi}{4}\right)$.

(b) $f(x)$ is monotonic decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(c) $\exists$ some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$.

(d) The equation $f(x) = 0$ has two roots in $\left[0, \frac{\pi}{2}\right]$.

9. Let $I_n = \int_0^{\sqrt{3}} \frac{dx}{1+x^n}$ (n = 1, 2, 3, ...) and $\lim_{n\to\infty} = I_0$ (say), then which of the following statement(s) is/are correct?

(a) $I_1 > I_0$ (b) $I_2 < I_0$

(c) $I_0 + I_1 + I_2 > 3$ (d) $I_0 + I_1 > 2$

10. A line L passing through the point $P(1, 4, 3)$, is perpendicular to both the lines $\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{4}$ and $\frac{x+2}{3} = \frac{y-4}{2} = \frac{z+1}{-2}$. If the position vector of point Q on L is (a_1, a_2, a_3) such that $(PQ)^2 = 357$, then $(a_1 + a_2 + a_3)$ can be

(a) 16 (b) 15 (c) 2 (d) 1

11. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

(a) $\angle QPR = 45°$

(b) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(c) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(d) The area of the circumcircle of the triangle PQR is 100π

12. Triangles $A_1A_2A_3$ and $B_1B_2B_3$ have side lengths (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively satisfying the relation $\sqrt{a_1+a_2+a_3}\sqrt{b_1+b_2+b_3} = \sqrt{a_1b_1} + \sqrt{a_2b_2} + \sqrt{a_3b_3}$, then which one of the following statements is/are true ?

(a) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ (b) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = 1$

(c) $\Delta A_1A_2A_3$ and $\Delta B_1B_2B_3$ are similar

(d) $\Delta A_1A_2A_3$ and $\Delta B_1B_2B_3$ are congruent

NUMERICAL VALUE TYPE

13. If α, β are roots of the equation $p(x^2 - x) + x + 5 = 0$ and p_1, p_2 are two values of p for which the roots α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then the value of $\frac{p_1}{p_2} + \frac{p_2}{p_1} =$ ________.

14. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is ______ .

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume values in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

15. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ______.

16. For each positive integer n,

Let $y_n = \frac{1}{n}((n+1)(n+2)\dots(n+n))^{\frac{1}{n}}$

For $x \in R$, let $[x]$ be the greatest integer less than or equal to x. If $\lim\limits_{n\to\infty} y_n = L$, then the value of $[L]$ is ____ .

17. $\lim\limits_{x\to 0} \dfrac{\tan[e^2]x^4 - \tan[-e^2]x^4}{\sin^4 x} =$ _______, [where $[\cdot]$ is G.I.F.]

18. Let $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix}$

and $C_r = \begin{bmatrix} r\cdot 3^r & 2^r \\ 0 & (r-1)3^r \end{bmatrix}$

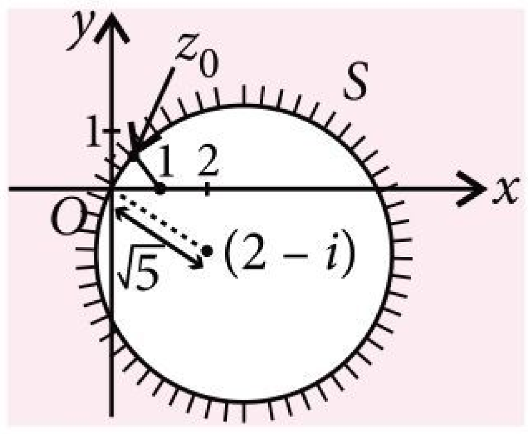
be given matrices. If $\sum\limits_{r=1}^{50} tr((AB)^r C_r) = 3 + a\cdot 3^b$ where $tr(A)$ denotes trace of matrix A, then find the value of $\frac{1}{20}(a+b)$. [Where a and b are relatively prime]

SOLUTIONS

PAPER-1

1. (c) : $\dfrac{1}{|z-1|}$, $z \in S$, attains its maximum when $|z-1|$ attains its minimum.

Also, the curve of $|z-2+i| \geq \sqrt{5}$ looks like the given figure.



Consider $\dfrac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i} = \dfrac{4-2\operatorname{Re}z_0}{2i\operatorname{Im}z_0+2i} = \dfrac{2(2-\operatorname{Re}z_0)}{2i(\operatorname{Im}z_0+1)}$

$= -i\lambda$, where $\lambda > 0$

Since $2 - \operatorname{Re} z_0 > 0$ and $\operatorname{Im} z_0 > 0$

$\therefore$ The principal argument of $\dfrac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is $-\pi/2$.

2. (d) : Let $\tan x_1$ and $\tan x_2$ are the roots of the equation $(1+k)\tan^2 x - 4\tan x + (k-1) = 0$.

$\therefore \quad \tan x_1 + \tan x_2 = \dfrac{4}{1+k}$ and $\tan x_1 \cdot \tan x_2 = \dfrac{k-1}{1+k}$

$$\Rightarrow \tan(x_1+x_2) = \frac{\tan x_1 + \tan x_2}{1-\tan x_1 \tan x_2} = \frac{\frac{4}{1+k}}{1-\left(\frac{k-1}{1+k}\right)} = 2$$

Since, the equation has real roots.

$\therefore \quad D \geq 0 \Rightarrow 16 - 4(k^2-1) \geq 0 \Rightarrow k^2 \leq 5$

3. (b) : $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ (for each $x \neq 0$) ...(i)

Replacing x by $\frac{1}{x}$ in (i), we get

$af\left(\frac{1}{x}\right) + bf(x) = x - 5$ ...(ii)

Eliminating $f\left(\frac{1}{x}\right)$ from (i) and (ii), we get

$(a^2-b^2)f(x) = \frac{a}{x} - bx - 5a + 5b$

$$\Rightarrow (a^2-b^2)\int_1^2 f(x)dx = \left[\left(a\log|x| - \frac{b}{2}x^2 - 5(a-b)x\right)\right]_1^2$$

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

			5				2					2	6		7	9		8		
7	1														3				4	
	9		8			4						5								
6	3		4	9	2							1	3					5		
	2					3		9							4		7			
						7									6			3		
				7		9		4			2	6	5	1		8		2		
2		4		8		6							7						6	
5												3	4			7				
									8	4			2							
								3				5								
										6										
	7					4	6						8		9			6	5	2
	3			9				8	3		9	7		4		8				
		9	5				1					2						7		
		2				8	4					8							7	4
	4			6											6			8		
			3				5									1				
3					1		7									4				7
				7	4	1							3					5	2	
5	1						8						5	7	1					

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

$$= a\log 2 - 2b - 10(a-b) - a\log 1 + \frac{b}{2} + 5(a-b)$$

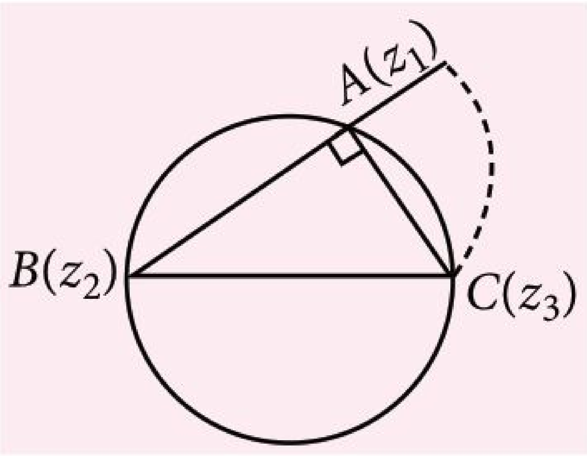
$$= a\log 2 - 5a + \frac{7}{2}b$$

$$\Rightarrow \int_1^2 f(x)dx = \frac{1}{a^2-b^2}\left[a\log 2 - 5a + \frac{7}{2}b\right]$$

4. (a) : $\because |z_1 + z_2| = ||z_1| - |z_2||$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pm\pi$$

and $|z_1 + i(z_3 - z_1)|$
$= |z_1| + |z_3 - z_1|$



$$\Rightarrow \arg\left(\frac{z_1}{z_3 - z_1}\right) = \frac{\pi}{2}$$

$\therefore$ Centre of circle is $\left(\frac{z_2 + z_3}{2}\right)$

and ΔABC is right angled triangle with $\angle A = 90°$

5. (b) : Consider three vectors

$\vec{n}_1 = (x_1, x_2, x_3), \vec{n}_2 = (y_1, y_2, y_3)$ and $\vec{n}_3 = (1,1,1)$.

From the given data,

$$\vec{n}_1 \cdot \vec{n}_2 = 0, \vec{n}_2 \cdot \vec{n}_3 = 0 \text{ and } \vec{n}_1 \cdot \vec{n}_3 = 0$$

i.e., $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ are mutually $\perp^r$ vectors.

Now, $\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2}, \frac{y_1^2}{y_1^2 + y_2^2 + y_3^2}$ and $\frac{1}{3}$ are the squares of the projections of the vector (1, 0, 0) on to the vectors of $\vec{n}_1, \vec{n}_2, \vec{n}_3$ respectively and hence their sum = 1

i.e., $\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2} + \frac{y_1^2}{y_1^2 + y_2^2 + y_3^2} + \frac{1}{3} = 1$

6. (a) : On simplifying, we have

$$\sqrt{3}\cos A \sin C = 2 - \sin A$$

$$\Rightarrow \sqrt{3}\cos A \sin C + \sin A = 2$$

As $\sin C \le 1$ and maximum value of $\sqrt{3}\cos A + \sin A = 2$

So, $\sin C = 1$, *i.e.*, $C = 90°$, $A = 30°$, $B = 60°$

So, maximum area = $\frac{1}{2}\cdot 2\cdot 2\sqrt{3} = 2\sqrt{3}$ sq. units

7. (a, c, d) : We have,

$$f(x) = \begin{cases} (x+1)^5 - 2x, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3. \end{cases}$$

Now, $f'(x) = \begin{cases} 5(x+1)^4 - 2, & x < 0 \\ 2x - 1, & 0 < x < 1 \\ 2x^2 - 8x + 7, & 1 < x < 3 \\ \log_e(x-2), & x > 3 \end{cases}$

and $f''(x) = \begin{cases} 2, & 0 < x < 1 \\ 4x - 8, & 1 < x < 3 \end{cases}$

Clearly $\lim\limits_{x\to 1^-} f''(x) \ne \lim\limits_{x\to 1^+} f''(x)$

and $f''(x)$ changes sign from positive to negative.

$\therefore f'(x)$ has a local maxima at $x = 1$ and $f'(x)$ is not differentiable at $x = 1$.

$\because \ f'(-1) < 0$. $\therefore f$ is not increasing on $(-\infty, 0)$

For $x < 0$, $\lim\limits_{x\to -\infty} f(x) = -\infty$, $\lim\limits_{x\to 0^-} f(x) = 1$

$\therefore$ The range of f in $(-\infty, 0)$ is $(-\infty, 1)$.

For $3 < x < \infty$, $f'(x) > 0$. $\therefore f$ is increasing and $f(3) = \frac{1}{3}$.

So, the range of f contains $\left(\frac{1}{3}, \infty\right)$.

Hence, the range of f is whole of R.

8. (a, b, c, d) : Given equation is

$(x^2 + 1)^2 = x(4x^2 + 5x + 4)$ ...(i)

$\Rightarrow x^4 - 4x^3 - 3x^2 - 4x + 1 = 0$

$$\Rightarrow x^2\left(x^2 - 4x - 3 - \frac{4}{x} + \frac{1}{x^2}\right) = 0$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) - 3 = 0$$

(as $x = 0$ does not satisfy (i))

$$\Rightarrow t^2 - 4t - 5 = 0, \text{ where } t = x + \frac{1}{x}$$

$\Rightarrow (t + 1)(t - 5) = 0$

$$\Rightarrow \left(x + \frac{1}{x} + 1\right)\left(x + \frac{1}{x} - 5\right) = 0$$

$\Rightarrow (x^2 + x + 1)(x^2 - 5x + 1) = 0$

$\Rightarrow x^2 + x + 1 = 0,\ x^2 - 5x + 1 = 0$

$$\Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2}, x = \frac{5 \pm \sqrt{21}}{2}$$

$$x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = C \pm iD,\ x = \frac{5}{2} \pm \sqrt{\frac{21}{4}} = A \pm \sqrt{B}$$

$$\Rightarrow C = \frac{-1}{2}, D = \frac{\sqrt{3}}{2},\ A = \frac{5}{2},\ B = \frac{21}{4}$$

9. (a, b, c, d) : Let E_1 and E_2 be the events of selecting bag A and B, respectively. Then,

$$P(E_1) = P(E_2) = \frac{1}{2},$$

$$P\left(\frac{\text{red}}{E_1}\right) = \frac{3}{5}, P\left(\frac{\text{red}}{E_2}\right) = \frac{7}{11}, P\left(\frac{\text{white}}{E_1}\right) = \frac{2}{5}$$

and $P\left(\frac{\text{white}}{E_2}\right) = \frac{4}{11}$

$$\therefore \quad P\left(\frac{E_2}{\text{red}}\right) = \frac{P(E_2)P\left(\frac{\text{red}}{E_2}\right)}{P(E_1)P\left(\frac{\text{red}}{E_1}\right) + P(E_2)P\left(\frac{\text{red}}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\cdot\frac{7}{11}}{\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{7}{11}} = \frac{35}{68}$$

$$\therefore \quad P\left(\frac{E_1}{\text{red}}\right) = 1 - P\left(\frac{E_2}{\text{red}}\right) = 1 - \frac{35}{68} = \frac{33}{68}$$

Now, $P\left(\frac{E_1}{\text{white}}\right) = \dfrac{P(E_1)P\left(\frac{\text{white}}{E_1}\right)}{P(E_1)P\left(\frac{\text{white}}{E_1}\right) + P(E_2)P\left(\frac{\text{white}}{E_2}\right)}$

$$= \frac{\frac{2}{5}}{\frac{2}{5} + \frac{4}{11}} = \frac{11}{21}$$

$$\therefore \quad P\left(\frac{E_2}{\text{white}}\right) = 1 - P\left(\frac{E_1}{\text{white}}\right) = 1 - \frac{11}{21} = \frac{10}{21}$$

10. (b, c, d) : Let $\tan^2\alpha = a$ and $\tan^2\beta = b$, then the given expression reduces to

$$\frac{(a+1)^2}{b} + \frac{(b+1)^2}{a} = 2\left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{a^2}{b} + \frac{b^2}{a} + \frac{1}{a} + \frac{1}{b}\right)$$

$$\geq 2(2) + 4(1) = 8$$

11. (c) : $\int_0^x \sqrt{1-(f'(t))^2}\, dt = \int_0^x f(t)dt$

Differentiating, we get $\sqrt{1-(f'(x))^2} = f(x)$

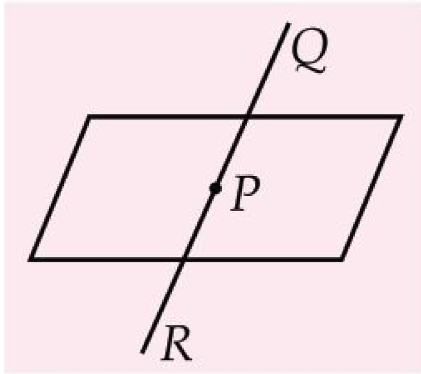
$$\Rightarrow \quad 1-(f'(x))^2 = (f(x))^2 \Rightarrow f'(x) = \sqrt{1-(f(x))^2}$$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{1-t^2}} = \int dx \Rightarrow \sin^{-1}(f(x)) = x + c$$

$$\Rightarrow \quad f(x) = \sin(x + c), f(0) = 0 \Rightarrow c = 0 \quad \therefore \quad f(x) = \sin x$$

Now, $f\left(\frac{1}{2}\right) = \sin\frac{1}{2} < \frac{1}{2}, f\left(\frac{1}{3}\right) = \sin\frac{1}{3} < \frac{1}{3}.$

12. (a) : Clearly, the equation of line QR be



$$\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1}$$

Let the point P be given as

$$P(2 + t, 3 + 4t, 5 + t)$$

As P lies in the plane $5x - 4y - z = 1$

$$\therefore \quad 5(2 + t) - 4(3 + 4t) - (5 + t) = 1$$

$$\Rightarrow \quad 10 + 5t - 12 - 16t - 5 - t = 1 \Rightarrow -12t - 7 = 1$$

$$\Rightarrow \quad 12t = -8 \quad \therefore \quad t = -\frac{8}{12} = -\frac{2}{3}$$

The co-ordinates of P are $\left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

Again let $S \equiv (2 + m, 3 + 4m, 5 + m)$, m being a constant.

As S is the foot of perpendicular, we have

$$1(2 + m - 2) + 4(3 + 4m - 1) + 1(5 + m - 4) = 0$$

$$\Rightarrow \quad m + 4(4m + 2) + m + 1 = 0$$

$$\Rightarrow \quad 18m + 9 = 0 \quad \therefore \quad m = -\frac{1}{2}$$

Thus $S \equiv \left(\frac{3}{2}, 1, \frac{9}{2}\right)$

Now, by distance formula

$$PS = \sqrt{\left(\frac{3}{2} - \frac{4}{3}\right)^2 + \left(1 - \frac{1}{3}\right)^2 + \left(\frac{9}{2} - \frac{13}{3}\right)^2}$$

$$= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{6}\right)^2} = \sqrt{\frac{1}{36} + \frac{4}{9} + \frac{1}{36}} = \sqrt{\frac{9}{18}} = \frac{1}{\sqrt{2}}$$

13. (7) : Hyperbola is $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

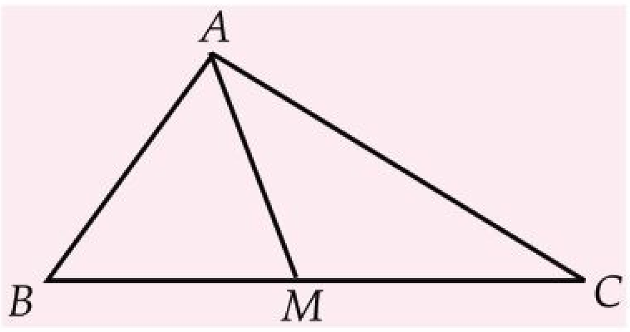
$$\therefore \quad a = \sqrt{\frac{144}{25}}, b = \sqrt{\frac{81}{25}} \text{ and } e_1 = \sqrt{1 + \frac{81}{144}} = \sqrt{\frac{225}{144}} = \frac{15}{12} = \frac{5}{4}$$

Now, foci $= (\pm ae_1, 0) = \left(\pm\frac{12}{5}\cdot\frac{5}{4}, 0\right) = (\pm 3, 0)$

Also, focus of ellipse $= (4e, 0)$.

$$\Rightarrow \quad e = \frac{3}{4}. \text{ Hence, } b^2 = 16\left(1 - \frac{9}{16}\right) = 7$$

14. (33) : $\overrightarrow{AM}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})$



$=\frac{1}{2}\{(3,\ 0,\ 4)+(5,\ -2,\ 4)\}$

$=\frac{1}{2}(8,\ -2,\ 8)=(4,\ -1,\ 4)$

$\therefore\quad |\overrightarrow{AM}|=\sqrt{4^2+1^2+4^2}=\sqrt{33}$

15. (2) : $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x(\tan 4x)}$

$=\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2\cdot\left(\dfrac{\tan 4x}{x}\right)}(3+\cos x)$

$=\lim\limits_{x\to 0}\left(\dfrac{2\sin^2 x}{x^2}\right)\cdot\left(\dfrac{x}{\tan 4x}\right)\ (3+\cos x)=2\times\dfrac{1}{4}\times 4=2$

16. (122) : $\dfrac{n(n+1)}{2}+k=2013$ and $n\geq k$

i.e. $n\geq 2013-\dfrac{n(n+1)}{2}$

Solving, $n\in(61.9,\ 62.9)$

So, n = 62 (integral) and k = 60

17. (11) : Number of students who gave wrong answers to atleast i questions = 2^{n-i}

Number of students who gave wrong answers to atleast n questions = 2^0 = 1

Number of students gave wrong answers to exactly i questions = $2^{n-i}-2^{n-(i+1)}$

Number of wrong answers

$=\sum\limits_{i=1}^{n-1} i\left(2^{n-i}-2^{n-(i+1)}\right)+(n)=2047$

$\Rightarrow\quad 1+2^1+2^2+\ldots+2^{n-1}=2047$

$\Rightarrow\quad 2^n=2048\quad\Rightarrow\quad n=11$

18. (6) : The given circles are $x^2+y^2=12$ and $x^2+y^2-5x+3y-2=0$

Common chord say, AB is $5x-3y-10=0$. Let the coordinates of point of intersection of tangents be $P(\alpha,\beta)$.

Equation of chord of contact of two tangents, drawn $P(\alpha,\beta)$, with respect to $x^2+y^2=12$ is $x\alpha+y\beta-12=0$.

Comparing the coefficients of common chord AB and chord of contact, we get $\dfrac{\alpha}{5}=\dfrac{\beta}{-3}=\dfrac{-12}{-10}\ \Rightarrow\alpha=6$

$\therefore\quad x$-coordinate is 6.

PAPER-2

1. (6) : Here, $n(S)$ = The number of ways in which papers of 5 students can be checked by the 5 teachers = 5^5

and $n(A)$ = choosing two teachers out of 5 × the number of ways in which 5 papers can be checked by exactly 2 teachers = ${}^5C_2\times(2^5-2)=300$

$\therefore$ Required probability

$=\dfrac{n(A)}{n(S)}=\dfrac{300}{5^5}=\dfrac{12}{125}=m$ (given)

Hence, $\dfrac{125m}{2}=6$

2. (9) : We have, $Q=\begin{vmatrix}2^2a_{11} & 2^3a_{12} & 2^4a_{13}\\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23}\\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix}$

$$= 2^9 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2 a_{31} & 2^2 a_{32} & 2^2 a_{33} \end{vmatrix} = 2^9 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= 2^{12}\det P = 2^{12}\cdot 2 = 2^{13} = 2^{k+4} \Rightarrow k = 9$

3. (4) : Putting $x = \omega, \omega^2$, we get

$(3 + \omega + \omega^2)^{2010} = a_0 + a_1\omega + a_2\omega^2 + ...$

$\Rightarrow\ 2^{2010} = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + ...$ ...(i)

and $2^{2010} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + ...$ ...(ii)

Adding (i) and (ii), we get

$$2^{2011} = 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 - ...$$

$$\therefore\ a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 + a_6 - ... = 2^{2010}$$

$= (2^2)^{1005} = k^{1005}$ (Given) $\Rightarrow k = 4$

4. (1) : $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x) \Rightarrow p = 7$ (given)

Again $\dfrac{u(x)}{v(x)} = 7 \Rightarrow \left(\dfrac{u(x)}{v(x)}\right)' = 0 \Rightarrow q = 0.$

Now, $\dfrac{p+q}{p-q} = \dfrac{7+0}{7-0} = 1$

5. (3) : Here $\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b}$

Also, $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c} = 2\cos\alpha$

We have, $\vec{c}\cdot\vec{a} = x\vec{a}\cdot\vec{a} + 0 + 0$

$\therefore\ x = \vec{c}\cdot\vec{a} = 2\cos\alpha$. Similarly, $y = 2\cos\alpha$

Now, as $c^2 = x^2 + y^2 + 1$

$\therefore$ We have $4 = 2(4\cos^2\alpha) + 1 \Rightarrow 3 = 8\cos^2\alpha$

6. (6) : We know, $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)f(h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x)[1 + h\,g(h)H(h) - 1]}{h}$$

$$= f(x)\lim_{h\to 0} g(h)\cdot H(h) = 6f(x)$$

7. (b, d) : Domain of $f = R$, Domain $g = R - [-1, 0)$; because $-1 \le x < 0 \Rightarrow 1 + [x] = 0$

$\therefore$ Domain of $f - g = R - [-1, 0)$

Since $e^{-x} > 0 \Rightarrow (1 + [x])y > 0$

It can be observed that $y > 0 \Rightarrow 1 + [x] > 0$,

or $y < 0 \Rightarrow 1 + [x] < 0$

$\therefore$ Range $g = R - \{0\}$

8. (c, d) : We have, $f'(x) = 5\sin^4 x\cos x - 5\cos^4 x\sin x = 5\sin x\cos x(\sin x - \cos x)(1 + \sin x\cos x)$

$\therefore\ f'(x) = 0$ at $x = \dfrac{\pi}{4}$. Also $f(0) = f\left(\dfrac{\pi}{2}\right) = 0$

Hence $\exists$ some $c \in \left(0, \dfrac{\pi}{2}\right)$ for which $f'(c) = 0$

(By Rolle's Theorem)

Also in $\left(0, \dfrac{\pi}{4}\right)$, $f(x)$ is decreasing and in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, $f(x)$ is increasing.

$\Rightarrow$ Minimum at $x = \dfrac{\pi}{4}$

As $f(0) = f\left(\dfrac{\pi}{2}\right) = 0 \Rightarrow$ 2 roots exists.

9. (a, c, d) : $I_1 = \ln(1 + \sqrt{3})$

$I_2 = \dfrac{\pi}{3}$

$$I_0 = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left[\int_0^1 \frac{dx}{1+x^n} + \underbrace{\int_1^{\sqrt{3}} \frac{dx}{1+x^n}}_{zero}\right] = \int_0^1 dx = 1.$$

Hence $I_0 = 1$.

10. (b, d) : Equation of the line passing through $P(1, 4, 3)$ and direction ratio's a, b & c is

$$\frac{x-1}{a} = \frac{y-4}{b} = \frac{z-3}{c} \quad ...(i)$$

Since (i) is perpendicular to $\dfrac{x-1}{2} = \dfrac{y+3}{1} = \dfrac{z-2}{4}$

and $\dfrac{x+2}{3} = \dfrac{y-4}{2} = \dfrac{z+1}{-2}$

Hence $2a + b + 4c = 0$ and $3a + 2b - 2c = 0$

$$\therefore\ \frac{a}{-2-8} = \frac{b}{12+4} = \frac{c}{4-3} \Rightarrow \frac{a}{-10} = \frac{b}{16} = \frac{c}{1}$$

Hence the equation of the line is

$$\frac{x-1}{-10} = \frac{y-4}{16} = \frac{z-3}{1} \quad ...(ii)$$

Now any point Q on (ii) can be taken as $(1 - 10\lambda, 16\lambda + 4, \lambda + 3)$

$\therefore$ Distance of Q from $P(1, 4, 3) = (10\lambda)^2 + (16\lambda)^2 + \lambda^2 = 357$

$\Rightarrow\ (100 + 256 + 1)\lambda^2 = 357$

$\Rightarrow\ \lambda = 1$ or -1

$\therefore\ Q$ is $(-9, 20, 4)$ or $(11, -12, 2)$

Hence, $a_1 + a_2 + a_3 = 15$ or 1

11. (b, c, d) : Using cosine rule,

$$\cos 30° = \frac{\sqrt{3}}{2} = \frac{(10\sqrt{3})^2 + 10^2 - PR^2}{2\cdot 10\cdot 10\sqrt{3}}$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cdot 2\times 10\sqrt{3}\times 10 = 400 - PR^2$$

$\Rightarrow$ $300 = 400 - PR^2$ $\Rightarrow PR^2 = 100$ $\therefore$ $PR = 10$

So, $QR = PR$ $\therefore$ $\angle QPR = 30°$

$$ar\,(\Delta PQR) = \frac{1}{2}\cdot 10\sqrt{3}\cdot 10\sin 30° = \frac{1}{2}\cdot 10\cdot 10\sqrt{3}\cdot\frac{1}{2} = 25\sqrt{3}$$

Also, $\angle QRP = 180° - 2\times 30° = 120°$

$$\text{Now, } r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{\frac{10+10+10\sqrt{3}}{2}} = \frac{25\sqrt{3}}{10+5\sqrt{3}} = \frac{25\sqrt{3}}{5(2+\sqrt{3})}$$

$$= 5\sqrt{3}(2-\sqrt{3}) = 10\sqrt{3} - 15$$

$$\text{Also, } R = \frac{10}{2\sin 30°} = 10$$

Area of circumcircle $= \pi R^2 = 100\pi$

12. (a, c) : We have, $\sqrt{(a_1+a_2+a_3)(b_1+b_2+b_3)}$

$$= \sqrt{a_1b_1} + \sqrt{a_2b_2} + \sqrt{a_3b_3}$$

Let $a_1 = P_1^2,\ a_2 = P_2^2,\ a_3 = P_3^2$

and $b_1 = Q_1^2,\ b_2 = Q_2^2,\ b_3 = Q_3^2$

$$\Rightarrow (P_1^2+P_2^2+P_3^2)(Q_1^2+Q_2^2+Q_3^2) = (P_1Q_1+P_2Q_2+P_3Q_3)^2$$

$$\Rightarrow (P_1Q_2-P_2Q_1)^2 + (P_2Q_3-P_3Q_2)^2 + (P_3Q_1-P_1Q_3)^2 = 0$$

$$\Rightarrow \frac{P_1}{Q_1} = \frac{P_2}{Q_2} = \frac{P_3}{Q_3} = \lambda$$

$$\text{or } \frac{\sqrt{a_1}}{\sqrt{b_1}} = \frac{\sqrt{a_2}}{\sqrt{b_2}} = \frac{\sqrt{a_3}}{\sqrt{b_3}} \quad \therefore \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

13. (254) : Here the given equation is

$p(x^2 - x) + x + 5 = 0$ $\Rightarrow$ $px^2 - (p-1)x + 5 = 0$

$$\therefore\ \alpha+\beta = \frac{p-1}{p} \text{ and } \alpha\beta = \frac{5}{p}$$

$$\text{Now } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5} \Rightarrow \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{4}{5}$$

$$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5} \Rightarrow \frac{(p-1)^2 - 10p}{5p} = \frac{4}{5}$$

$$\Rightarrow p^2 - 16p + 1 = 0 \Rightarrow p_1 + p_2 = 16 \text{ and } p_1p_2 = 1$$

$$\text{Now, } \frac{p_1}{p_2} + \frac{p_2}{p_1} = \frac{(p_1+p_2)^2 - 2p_1p_2}{p_1p_2} = \frac{256-2}{1} = 254$$

14. (2) : Let $f(x) = \sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i$

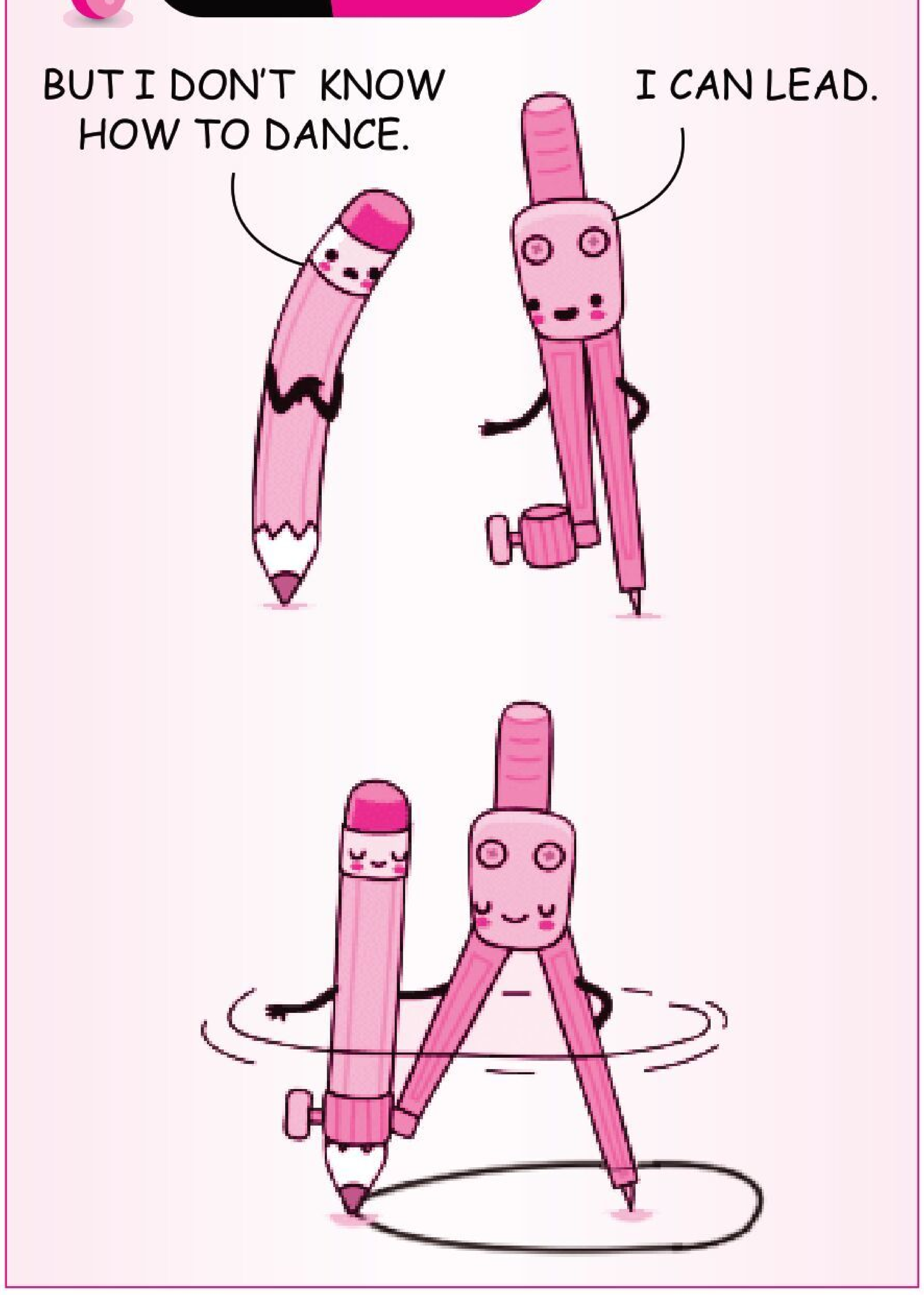
$$= (x^2 + x^3 + \text{.... to } \infty) - x\left(\frac{x}{2} + \left(\frac{x}{2}\right)^2 + \text{.... to } \infty\right)$$

$$= \frac{x^2}{1-x} - \frac{x\cdot\frac{x}{2}}{1-\frac{x}{2}} = \frac{x^2}{1-x} - \frac{x^2}{2-x}$$

Again, let $g(x) = \sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$

$$= \left[\left(\frac{-x}{2}\right) + \left(\frac{-x}{2}\right)^2 + \text{... to } \infty\right] - \left[(-x) + (-x)^2 + \text{... to } \infty\right]$$

COMIC CAPSULE



$$=\frac{-\frac{x}{2}}{1+\frac{x}{2}}-\frac{-x}{1+x}=\frac{x}{1+x}-\frac{x}{2+x}$$

The given equation becomes

$\sin^{-1}f(x) + \cos^{-1}g(x) = \pi/2$

So we must have $f(x) = g(x)$

$$\Rightarrow \frac{x^2}{1-x}-\frac{x^2}{2-x}=\frac{x}{1+x}-\frac{x}{2+x}$$

$$\Rightarrow \frac{x^2}{(1-x)(2-x)}=\frac{x}{(2+x)(1+x)}$$

$\Rightarrow$ $x = 0$ and $x(2 + x)(1 + x) = (1 - x)(2 - x)$

Let $h(x) = x(2 + x)(1 + x) - (1 - x)(2 - x)$

$= x^3 + 2x^2 + 5x - 2$

$$\text{Now, } h\left(-\frac{1}{2}\right)=-\frac{1}{2}\left(2-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)-\left(1+\frac{1}{2}\right)\left(2+\frac{1}{2}\right)$$

$$=-\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}-\frac{3}{2}\cdot\frac{5}{2}=\frac{3}{2}\left(-\frac{1}{4}-\frac{5}{2}\right)<0$$

$$h\left(\frac{1}{2}\right)=\frac{1}{2}\left(2+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)-\left(1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)$$

$$=\frac{1}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}-\frac{1}{2}\cdot\frac{3}{2}>0$$

$\therefore$ $\exists$ a root between $-\frac{1}{2}$ and $\frac{1}{2}$.

Also, $h'(x) = 3x^2 + 4x + 5 > 0$

$\Rightarrow$ $h(x)$ has exactly one real root in $\left(-\frac{1}{2},\frac{1}{2}\right)$.

15. (3748) : Here $X = \{1, 6, 11, ..., 10086]$

and $Y = \{9, 16, 23, ..., 14128\}$

The intersection of X and Y is an A.P. with 16 as first term and 35 as common difference.

The sequence becomes 16, 51, 86, ...

Now, k^{th} term $= 16 + (k - 1)\ 35 \le 10086$

i.e. $k\le\frac{10105}{35}$ $\therefore$ $k \le 288$ (As k is to be an integer)

Hence, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= 2018 + 2018 - 288 = 3748$

16. (1) : We have

$$\log y_n=\frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+...+\log\left(1+\frac{x}{n}\right)\right\}$$

$$\therefore\ \lim_{n\to\infty}\log y_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$=\int_0^1\log(1+x)dx=\int_1^2\log x\,dx$$

$$=\left[x\log x-x\right]_1^2=2\log 2-1=\log\frac{4}{e}$$

$\therefore$ $L = 4/e$ $\Rightarrow$ $[L] = 1$.

17. (15) : $\displaystyle\lim_{x\to 0}\frac{\tan[e^2]x^4-\tan[-e^2]x^4}{\sin^4 x}$

$$=\lim_{x\to 0}\frac{\tan 7x^4-\tan(-8)x^4}{\sin^4 x}=\lim_{x\to 0}\frac{\tan 7x^4+\tan 8x^4}{\sin^4 x}$$

$$=\frac{7\lim\limits_{x\to 0}\frac{\tan 7x^4}{7x^4}+8\lim\limits_{x\to 0}\frac{\tan 8x^4}{8x^4}}{\lim\limits_{x\to 0}\frac{\sin^4 x}{x^4}}=\frac{7+8}{1}=15$$

18. (5) : $AB=\begin{bmatrix}1 & \frac{3}{2}\\ 1 & 2\end{bmatrix}\begin{bmatrix}4 & -3\\ -2 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

$\therefore$ $(AB)^1C_1 = C_1$, $(AB)^2C_2 = C_2$ and so on.

Also, $tr(C_r) = r\cdot 3^r + (r - 1)\cdot 3^r = (2r - 1)\cdot 3^r$

$$\text{Now, }\sum_{r=1}^{50}tr((AB)^r C_r)=tr((AB)^1C_1)+tr((AB)^2C_2)$$
$$+........+tr((AB)^{50}C_{50})=S\ \text{(Let)}$$

$\therefore$ $S = tr\,(C_1) + tr\,(C_2) + ... + tr\,(C_{50})$

$S = 1\cdot 3^1 + 3\cdot 3^2 + 5\cdot 3^3 + ... + 99\cdot 3^{50}$

$3S = 1\cdot 3^2 + 3\cdot 3^3 + ... + 97\cdot 3^{50} + 99\cdot 3^{51}$

Subtracting above equations, we get

$-2S = 1\cdot 3 + 2\cdot 3^2 + 2\cdot 3^3 + ... + 2\cdot 3^{50} - 99\cdot 3^{51}$

$= -3 + 2\cdot 3 + 2\cdot 3^2 + ... + 2\cdot 3^{50} - 99\cdot 3^{51}$

$$= -3 + 2\cdot\frac{3\cdot(3^{50}-1)}{3-1} - 99\cdot 3^{51} = -3 + 3^{51} - 3 - 99\cdot 3^{51}$$

$= -6 - 98\cdot 3^{51}$ $\Rightarrow$ $S = 3 + 49\cdot 3^{51}$

$\therefore$ $a + b = 100$

Hence $\frac{1}{20}(a+b)=5$

Challenging PROBLEMS ON JEE

Single Option Correct Type

1. If the sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle is maximum, then the angle between these is

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

2. If n arithmetic means are inserted between two sets of numbers a, $2b$ and $2a$, b, where $a, b \in R$. Suppose that m^{th} arithmetic means between these two sets of numbers is same, then the ratio $a : b$ equals to

(a) $(n - m + 1) : m$ (b) $(n - m + 1) : n$
(c) $m : (n - m + 1)$ (d) $n : (n - m + 1)$

3. The cartesian equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r}\cdot(\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$ is

(a) $3x - 4y + 4z = 5$
(b) $x - 2y + 4z = 3$
(c) $5x - 2y - 12z + 47 = 0$
(d) $2x + 3y + 4 = 0$

4. In a three dimensional co-ordinate system P, Q and R are images of a point $A(a, b, c)$ in the xy, yz and zx planes respectively. If G is the centroid of triangle PQR, then area of triangle AOG is (O is the origin)

(a) 0 (b) $a^2 + b^2 + c^2$
(c) $\frac{2}{3}(a^2 + b^2 + c^2)$ (d) None of these

5. Two natural numbers a and b are selected at random. The probability that $a^2 + b^2$ is divisible by 7 is

(a) 3/8 (b) 1/7 (c) 3/49 (d) 1/49

6. The number of positive integral solutions of the equation $\begin{vmatrix} y^3+1 & y^2z & y^2x \\ yz^2 & z^3+1 & z^2x \\ yx^2 & x^2z & x^3+1 \end{vmatrix} = 11$ is

(a) 1 (b) 2 (c) 3 (d) 4

More Than One Option Correct Type

7. Let P be the point on the parabola $y^2 = 8x$ which is at the least distance from the centre C of the circle $x^2 + y^2 - 8x - 32y + 256 = 0$. If Q be the point on the circle dividing the line segment CP internally, then

(a) x intercept of the normal at P is 12.
(b) length of $QP = 4(\sqrt{5} - 1)$.
(c) equation of tangent at P is $x - 2y + 8 = 0$.
(d) slope of tangent to the circle at $Q = 1/2$.

8. If $2x - y + 1 = 0$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following cannot be the sides of a right angled triangle?

(a) a, 4, 2 (b) a, 4, 1
(c) $2a$, 8, 1 (d) $2a$, 4, 1

9. The eccentric angle of a point on the ellipse $3x^2 + 5y^2 = 15$ at distance 2 units from the origin is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$
(c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$

10. If $\cos\theta + \cos\phi = \alpha$, $\cos 2\theta + \cos 2\phi = \beta$ and $\cos 3\theta + \cos 3\phi = \gamma$, then

(a) $\cos^2\theta + \cos^2\phi = 1 + \frac{\beta}{2}$
(b) $\cos\theta \cdot \cos\phi = \frac{\alpha^2}{2} - \frac{\beta+2}{4}$
(c) $2\alpha^3 + \gamma = 3\alpha(1 + \beta)$
(d) $\alpha + \beta + \gamma = 3\alpha\beta\gamma$

Comprehension Type

Paragraph for Q. No. 11 and 12

If a source of light is placed at the fixed point of a parabola and if the parabola is a reflecting surface, then the ray will bounce back in a line parallel to the axis of the parabola.

11. A ray of light is coming along the line $y = 2$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the x-y plane is a parabola $y^2 = 8x$, then the equation of the reflected ray is

(a) $2x + 5y = 4$ (b) $3x + 2y = 6$
(c) $4x + 3y = 8$ (d) $5x + 4y = 10$

12. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $y^2 + 10y - 4x + 17 = 0$. After reflection, the ray must pass through the point

(a) $(-2, -5)$ (b) $(-1, -5)$
(c) $(-3, -5)$ (d) $(-4, -5)$

Paragraph for Q. No. 13 and 14

If $S_1 = 0$ and $S_2 = 0$ are equations of circles intersecting in real and distinct points A and B, then AB is common chord of the circles $S_1 = 0$, $S_2 = 0$ known as radical axis of circles which is $\perp$ to line segment joining the centres of the circles. If $S_1 = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S_2 = x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then equation of radical axis is given by $2x(g - g') + 2y(f - f') + c - c' = 0$. If circles touching each other externally or internally, then point of contact can be determined by section formula. Consider the circles $S_1 = x^2 + y^2 - 4x - 2y + 4 = 0$ and $S_2 = x^2 + y^2 - 12x - 8y + 36 = 0$.

13. The point at which the circles $S_1 = 0$, $S_2 = 0$ touches each other is

(a) $\left(\frac{8}{5}, \frac{14}{5}\right)$ (b) $\left(\frac{14}{5}, \frac{8}{5}\right)$
(c) $\left(-\frac{8}{5}, \frac{14}{5}\right)$ (d) $\left(\frac{8}{5}, -\frac{14}{5}\right)$

14. The equation of direct common tangent to the circles $S_1 = 0$ and $S_2 = 0$ is

(a) $y = 2$ (b) $x = 0$
(c) $24x - 7y = 16$ (d) $24x + 7y = 16$

Matrix Match Type

15. Match the following :

	Column-I		Column-II
A.	If e be the eccentricity of the hyperbola $36x^2 + 288x - 64y^2 - 128y - 1792 = 0$, then the value of $4e$ equals	P.	7
B.	The number of values of K, such that the line $y = 4x + K$ touches the curve $x^2 + 4y^2 = 4$ is	Q.	3
C.	The eccentricity of an ellipse with centre at $O(0, 0)$ is $\frac{1}{3}$. If one directrix is $x = 6$ and ellipse reduces to $K_1x^2 + K_2y^2 = K_3$, then $\frac{K_3 - K_1 - K_2}{5} =$	R.	5
D.	If the foci of an ellipse and the hyperbola whose equations are respectively $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and $\frac{25x^2}{144} - \frac{25y^2}{81} = 1$, coincide with each other, then the value of b^2 equals	S.	2

Numerical Value Type

16. If $2\tan^2 x - 5\sec x = 1$ for exactly 7 distinct values of $x \in \left[0, \frac{n\pi}{2}\right], n \in N$, then the greatest value of n is _____.

17. From a point P outside a circle with centre at C, tangents PA and PB are drawn such that $\frac{1}{(CA)^2} + \frac{1}{(PA)^2} = \frac{1}{16}$, then the length of chord AB is ______.

18. If 7 divides $32^{32^{32}}$, the remainder is ______.

19. There exist positive integers A, B and C with no common factors greater than 1, such that $A \log_{200}5 + B \log_{200}2 = C$. The sum $A + B + C$ equals to ______.

20. The number of 4-digit numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 such that the least digit used is 4, when repetition of digits is allowed, is ______.

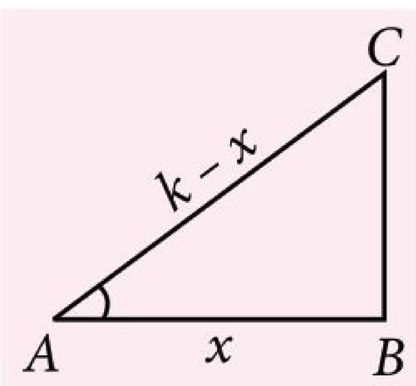
SOLUTIONS

1. (c) : $AB + AC = \text{constant} = k$

$\therefore\ BC^2 = (k - x)^2 - x^2 = k^2 - 2kx$

$$\therefore\ \Delta = \frac{1}{2} BC \cdot AB = \frac{1}{2} x\sqrt{k^2 - 2kx}$$

$$\text{Let } Z = \Delta^2 = \frac{1}{4}x^2(k^2 - 2kx)$$
$$= \frac{1}{4}(k^2x^2 - 2kx^3)$$



Z will be max, when $x = k/3$

Now, $\cos\theta = \dfrac{x}{k-x} = \dfrac{k/3}{k-k/3} = \dfrac{1}{2}$

$\therefore\ \ \theta = \pi/3$

2. (c) : Let $A_1, A_2, \ldots\ldots, A_n$ be arithmetic means between a and $2b$, then $A_m = a + m\left(\dfrac{2b-a}{n+1}\right)$

Again, let $B_1, B_2, \ldots\ldots, B_n$ be arithmetic means between $2a$ and b, then $B_m = 2a + m\left(\dfrac{b-2a}{n+1}\right)$

Now, $A_m = B_m$(Given)

$\Rightarrow a + m\left(\dfrac{2b-a}{n+1}\right) = 2a + m\left(\dfrac{b-2a}{n+1}\right)$

$\Rightarrow m\left(\dfrac{b+a}{n+1}\right) = a \Rightarrow \dfrac{a}{b} = \dfrac{m}{n-m+1}$

3. (c) : Equation of any plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r}\cdot(\hat{i} - \hat{j}) + 4 = 0$ is $2x - 3y + 4z - 1 + \lambda(x - y + 4) = 0$

or $(2 + \lambda)\,x - (3 + \lambda)y + 4z + 4\lambda - 1 = 0$

The plane is perpendicular to the plane

$\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$ if $2(2 + \lambda) + (3 + \lambda) + 4 = 0$

$\Rightarrow\ \ 11 + 3\lambda = 0\ \ \Rightarrow\ \ \lambda = -\,11/3$

The required equation of the plane is

$3(2x - 3y + 4z - 1) - 11(x - y + 4) = 0$

$\Rightarrow\ \ 5x - 2y - 12z + 47 = 0$

4. (a) : Point A is $(a, b, c) \Rightarrow$ Images of point A *i.e.*, P, Q and R are $(a, b, -c)$, $(-a, b, c)$ and $(a, -b, c)$ respectively.

$\therefore$ Centroid of triangle PQR is $\left(\dfrac{a}{3}, \dfrac{b}{3}, \dfrac{c}{3}\right)$

$\Rightarrow\ \ G \equiv \left(\dfrac{a}{3}, \dfrac{b}{3}, \dfrac{c}{3}\right)$

$\Rightarrow$ A, O, G are collinear $\therefore$ Area of triangle AOG is 0.

5. (d) : a, b are of the form

$a, b \in \{7m, 7m+1, 7m+2, 7m+3, 7m+4, 7m+5, 7m+6\}$

Now, $a^2, b^2 \in \{7m_1, 7m_1+1, 7m_1+4, 7m_1+2, 7m_1+2, 7m_1+4, 7m_1+1\}$

For required condition, a^2, b^2 must be of the form $7m$.

$\therefore$ Required probability $= \dfrac{1}{49}$

6. (c) : Multiply R_1, R_2 and R_3 by y, z and x respectively and then take common y, z and x from C_1, C_2, C_3 respectively, then

$$\begin{vmatrix} y^3+1 & y^3 & y^3 \\ z^3 & z^3+1 & z^3 \\ x^3 & x^3 & x^3+1 \end{vmatrix} = 11 \Rightarrow \begin{vmatrix} 1 & 0 & y^3 \\ -1 & 1 & z^3 \\ 0 & -1 & x^3+1 \end{vmatrix} = 11$$

(Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

$\Rightarrow\ \ 1(x^3 + 1 + z^3) + y^3(1) = 11 \Rightarrow x^3 + y^3 + z^3 = 10$

So, solutions are $(1, 1, 2)$, $(1, 2, 1)$ or $(2, 1, 1)$

7. (a, b, c, d) : Equation of circle is

$x^2 + y^2 - 8x - 32y + 256 = 0$

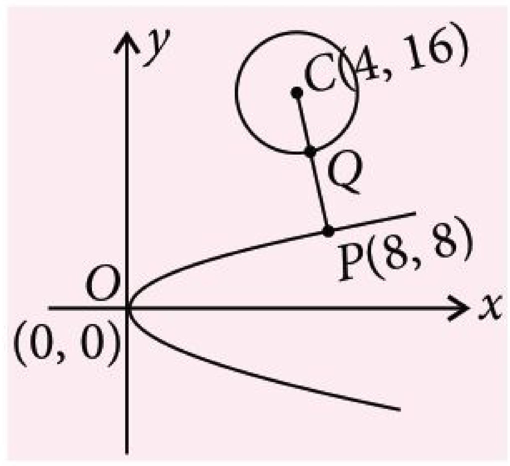
$\Rightarrow\ \ (x - 4)^2 + (y - 16)^2 = 4^2$

$\therefore\ \ $ Centre $C \equiv (4, 16)$ and radius $R = 4$

Now, equation of normal to the parabola $y^2 = 8x$ is

$y = mx - 4m - 2m^3$ ...(i)

For the least distance from the centre C to parabola the normal must pass through the centre $(4, 16)$ of circle



$\therefore\ \ 16 = 4m - 4m - 2m^3$

$\Rightarrow\ \ m^3 = (-2)^3 \Rightarrow\ \ m = -\,2$

$\therefore\ \ $ Equation of normal is $y = -2x + 24$ (Using (i))

$\therefore\ \ $ Slope of tangent at $Q = 1/2$.

Also, x intercept of normal at P is obtained by putting $y = 0$ in (i), $x = 12$

Now, any point on the parabola $y^2 = 4ax$ is $(am^2, -2am)$ $= (8, 8)$ (using $a = 2$, $m = -2$)

$\therefore\ \ $ Equation of tangent to parabola at P is

$y - 8 = \dfrac{1}{2}(x - 8)\ \Rightarrow x - 2y + 8 = 0$

Now $PQ = CP - CQ = CP -$ radius of circle (R)

$= 4\sqrt{5} - 4 = 4(\sqrt{5} - 1)$

8. (a, b, c) : The line $2x - y + 1 = 0$ be tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{16} = 1$ if both roots of the equation $\dfrac{x^2}{a^2} - \dfrac{(2x+1)^2}{16} = 1$ are equal.

i.e., discriminant of $(16 - 4a^2)x^2 - 4a^2x - 17a^2 = 0$

$\Rightarrow\ \ 16a^4 = 4(-17a^2)(16 - 4a^2)\ \Rightarrow\ \ a = \sqrt{17}/2$

Thus, we have set of numbers

$\left(\dfrac{\sqrt{17}}{2}, 4, 2\right)$; $\left(\dfrac{\sqrt{17}}{2}, 4, 1\right)$; $(\sqrt{17}, 8, 1)$ and $(\sqrt{17}, 4, 1)$

Only set $(\sqrt{17}, 4, 1)$ represents the sides of a right angled triangle.

9. (a, b, c, d) : We have, $\frac{x^2}{5}+\frac{y^2}{3}=1$ ...(i)

$\therefore$ The point (P) on the ellipse is $(\sqrt{5}\cos\theta, \sqrt{3}\sin\theta)$, where θ is eccentric angle.

Given, distance OP = 2 units (where O is origin)

$\therefore$ $5\cos^2\theta + 3\sin^2\theta = (2)^2 \Rightarrow \cos 2\theta = 0$

$\therefore$ $2\theta = (2n+1)\frac{\pi}{2},\ (n \in I) \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

10. (a, b, c): $(\cos\theta + \cos\phi)^2 = \alpha^2$ ...(i)

$\Rightarrow \cos^2\theta + \cos^2\phi + 2\cos\theta\cos\phi = \alpha^2$

Now, $\cos 2\theta + \cos 2\phi = \beta$

$\Rightarrow (2\cos^2\theta - 1) + (2\cos^2\phi - 1) = \beta$

$\Rightarrow 2(\cos^2\theta + \cos^2\phi) = \beta + 2$

$\Rightarrow \cos^2\theta + \cos^2\phi = \frac{\beta}{2} + 1$ ...(ii)

From (i) and (ii), we get $\cos\theta\cdot\cos\phi = \frac{\alpha^2}{2} - \frac{\beta+2}{4}$

Also, $\cos 3\theta + \cos 3\phi = \gamma$

$\Rightarrow (4\cos^3\theta - 3\cos\theta) + (4\cos^3\phi - 3\cos\phi) = \gamma$

$\Rightarrow 4(\cos^3\theta + \cos^3\phi) - 3(\cos\theta + \cos\phi) = \gamma$

$\Rightarrow 4[(\cos\theta + \cos\phi)(\cos^2\theta + \cos^2\phi - \cos\theta\cos\phi)]$
$-3(\cos\theta + \cos\phi) = \gamma$

$\Rightarrow 4\left[\alpha\left(\frac{\beta+2}{2} - \frac{1}{2}\left(\alpha^2 - \frac{(\beta+2)}{2}\right)\right)\right] - 3\alpha = \gamma$

$\therefore$ $2\alpha^3 + \gamma = 3\alpha(1+\beta)$

11. (c) : Point of intersection of $y = 2$ and $y^2 = 8x$ is $P\left(\frac{1}{2}, 2\right)$ and focus of the parabola is $S(2, 0)$

$\therefore$ Equation of the reflected ray is $y - 0 = \frac{2-0}{1/2-2}(x-2)$

$\Rightarrow y = -\frac{4}{3}(x-2) \Rightarrow 4x + 3y = 8$

12. (b) : $\because$ $y^2 + 10y - 4x + 17 = 0$

$\Rightarrow (y+5)^2 - 25 - 4x + 17 = 0$

$\Rightarrow (y+5)^2 = 4x + 8 \Rightarrow (y+5)^2 = 4(x+2)$

Let $y + 5 = Y$, $x + 2 = X$, then $Y^2 = 4X$

Focus is $X = 1$, $Y = 0$, *i.e.*, $(-1, -5)$

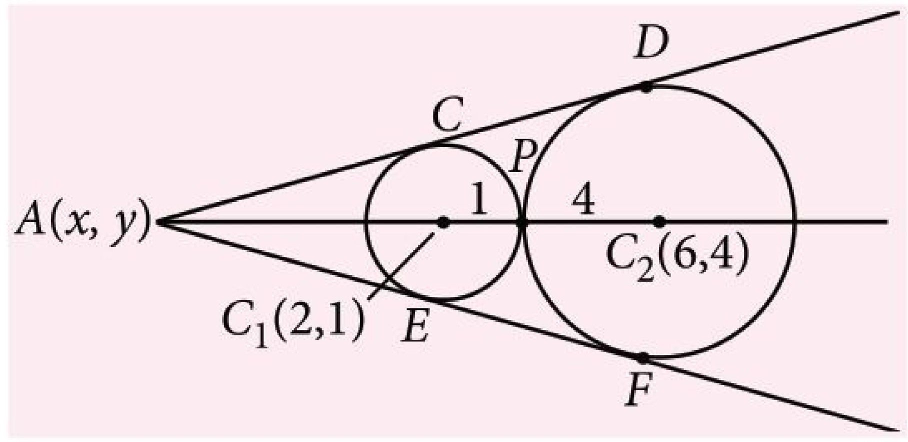
After reflection, the ray must pass through focus $(-1, -5)$

13. (b) : Given $S_1 = x^2 + y^2 - 4x - 2y + 4 = 0$

$\Rightarrow C_1(2, 1)$ and $r_1 = 1$

$S_2 = x^2 + y^2 - 12x - 8y + 36 = 0 \Rightarrow C_2(6, 4)$ and $r_2 = 4$

Now, $r_1 + r_2 = 1 + 4 = 5 = C_1C_2$ (distance between the centres of circles.



As $r_1 + r_2 = C_1C_2$ which means circles touch each other externally at a point say P which divides C_1C_2 in the ratio 1 : 4

The co-ordinates of P

$$= \left(\frac{1\times6+4\times2}{5}, \frac{1\times4+4\times1}{5}\right) = \left(\frac{14}{5}, \frac{8}{5}\right)$$

14. (c) : The direct common tangents meet (intersect) each other at $\left(\frac{2}{3}, 0\right)$, let its slope be m and its equation is given by $y - 0 = m\left(x - \frac{2}{3}\right) \Rightarrow y = mx - \frac{2m}{3}$

As direct common tangents are perpendicular to the lines joining the point of contact and centres of the circles and the distance from point of contact to centres is equal to the radii of circles.

$\therefore$ Distance from (2, 1) to the tangent line $mx - y - \frac{2m}{3} = 0$ is equal to 1.

$\therefore$ $\frac{2m - 1 - \frac{2m}{3}}{\sqrt{1+m^2}} = 1 \Rightarrow m = 0, m = \frac{24}{7}$

$\therefore$ Equations of direct common tangents to S_1 and S_2 are $y = 0$ and $y = \frac{24}{7}x - \frac{16}{7} \Rightarrow 24x - 7y = 16$.

15. A → R, B → S, C → Q, D → P

(A) $36x^2 + 288x - 64y^2 - 128y - 1792 = 0$

$\Rightarrow 36(x+4)^2 - 64(y+1)^2 = 2304$

$\Rightarrow \frac{(x+4)^2}{64} - \frac{(y+1)^2}{36} = 1 \Rightarrow a^2 = 64, b^2 = 36$

$\therefore$ $e = \sqrt{1 + \frac{36}{64}} = \frac{10}{8} = \frac{5}{4} \Rightarrow 4e = 5$

(B) Given curve is $x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$

It is an ellipse, where $a^2 = 4$, $b^2 = 1$ and the line $y = mx + K$ will be tangent if $K^2 = a^2m^2 + b^2$

$\therefore$ $K^2 = 4\cdot4^2 + 1 \Rightarrow K = \pm\sqrt{65}$

$\therefore$ Number of values of K = 2

(C) Equation of directrix is given by

$x = \frac{a}{e} \Rightarrow a = xe \Rightarrow a = 2$

Now, $b^2 = a^2(1 - e^2) = 4\left(1 - \frac{1}{9}\right) = \frac{4\times8}{9} = \frac{32}{9}$

$\therefore$ The equation of ellipse is $\frac{x^2}{4}+\frac{9y^2}{32}=1$

$\Rightarrow$ $8x^2 + 9y^2 = 32$ $\Rightarrow$ $K_1x^2 + K_2y^2 = K_3$

$\therefore$ $\frac{K_3 - K_1 - K_2}{5} = \frac{32-8-9}{5} = 3$

(D) For hyperbola, we have

$$e^2 = \frac{a^2+b^2}{a^2} = \frac{144+81}{144} = \frac{225}{144} \Rightarrow e = \frac{5}{4}$$

and $a^2 = \frac{144}{25} \Rightarrow a = \frac{12}{5}$

$\therefore$ Foci are $(\pm ae, 0) = \left(\pm\frac{12}{5}\times\frac{5}{4}, 0\right) = (\pm 3, 0)$

Now, for ellipse ($ae = 3$) (foci of an ellipse and hyperbola coincide with each other), $a = 4$

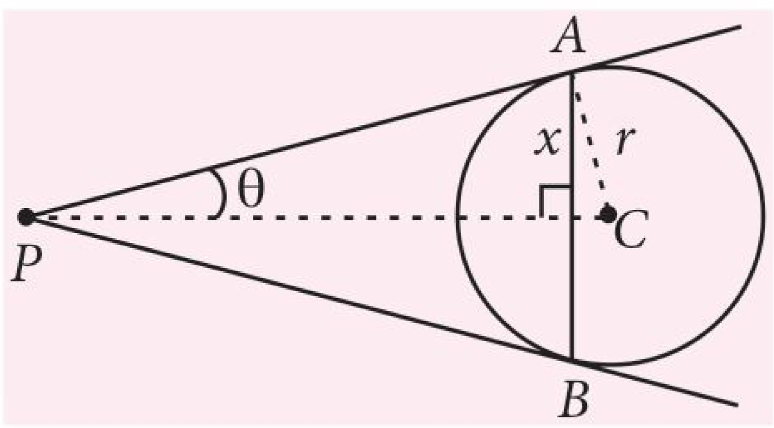
Now, $b^2 = a^2(1 - e^2) = a^2 - a^2e^2 = 16 - 9 = 7$

16. (15) : Now, $\sec x = 3 \Rightarrow \cos x = \frac{1}{3}$

It gives two values of x in each of $[0, 2\pi]$, $(2\pi, 4\pi]$, $(4\pi, 6\pi]$ and one value in $6\pi + \frac{3\pi}{2} = \frac{15\pi}{2}$

$\therefore$ Greatest value of $n = 15$

17. (8) : $\tan\theta = \frac{r}{PA}$



Given $\frac{1}{r^2}+\frac{1}{PA^2}=\frac{1}{16}$

$\Rightarrow$ $\frac{\cot^2\theta+1}{(PA)^2} = \frac{1}{16}$

$\Rightarrow$ $PA\sin\theta = x = 4$ $\therefore$ Length of Chord $AB = 2x = 8$.

18. (4) : $32 = 2^5 \Rightarrow (32)^{32} = (2^5)^{32}$

$= 2^{160} = (3-1)^{160} = 3m + 1,\ m \in N$

$\therefore$ $(32)^{32^{32}} = (32)^{3m+1} = 2^{5(3m+1)}$

$= 2^{3(5m+1)}2^2 = 4 \cdot 8^{5m+1} = 4(7+1)^{5m+1}$

$= 4(7n + 1),\ n \in N = 28n + 4$

$\therefore$ When 7 divides $(32)^{32^{32}}$, remainder = 4

19. (6) : Given $A \log_{200}5 + B \log_{200}2 = C$

$\Rightarrow$ $A \log 5 + B \log 2 = C \log 200 = C \log(5^2\, 2^3)$

$= 2C \log 5 + 3\,C \log 2$

Hence, $A = 2C$ and $B = 3C$

For no common factor greater than 1, $C = 1$

So, $A = 2$, $B = 3$. Thus $A + B + C = 6$

20. (671) : Least digit used = 4

$\therefore$ We can use 4, 5, 6, 7, 8, 9. But remember that at least one 4 must be used.

Now, 1st blank can be filled in 6 ways.

2nd blank can be filled in 6 ways.

3rd blank can be filled in 6 ways.

4th blank can be filled in 6 ways.

$\therefore$ 4 blanks can be filled in 6^4 ways. But out of these, some may contain no digit 4 at all. Let us find them. Each blank can be filled in 5 ways (by 5, 6, 7, 8, or 9)

$\therefore$ 4 blanks can be filled in 5^4 ways (no 4 at all)

$\therefore$ Required Number = $6^4 - 5^4$ (at least one 4) = 671.

beat the TIME TRAP

Duration : 30 minutes

SECTION-I

Single Option Correct Type

1. BC is latus rectum of a parabola $y^2 = 4ax$ and A is its vertex. The minimum length of projection of BC on a tangent drawn in portion BAC is

(a) $\sqrt{2}a$ (b) $2\sqrt{2}a$
(c) $2a$ (d) $3\sqrt{2}a$

2. Least value of the expression $\frac{1}{2bx-(x^2+b^2+\sin^2 x)}$, $x \in [-1, 0]$, $b \in [2, 3]$ is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$
(c) $-\frac{1}{8+\sin^2 1}$ (d) none of these

3. If in a right angled triangle ABC, $4\sin A \cos B - 1 = 0$ and $\tan A$ is real, then

(a) angles are in A.P. (b) angles are in G.P.
(c) angles are in H.P. (d) none of these

4. If $|z| = 2$ and $\frac{z_1 - z_3}{z_2 - z_3} = \frac{z-2}{z+2}$, then z_1, z_2 and z_3 will be vertices of a

(a) equilateral triangle
(b) acute angled triangle
(c) right angled triangle
(d) none of these

5. If $a^2 + b^2 - c^2 - 2ab = 0$, then the point(s) of concurrency of family of straight lines $ax + by + c = 0$ lie(s) on the line

(a) $y = x$ (b) $y = x + 1$
(c) $y = -x$ (d) $x + y = 1$

6. If $f''(x) > 0\ \forall\ x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \frac{\pi}{2}$, then $g(x)$ is increasing in

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$
(c) $\left(0, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

7. A vector $\vec{r}$ is equally inclined with the vectors $\vec{a} = \cos\theta\hat{i} + \sin\theta\hat{j}$, $\vec{b} = -\sin\theta\hat{i} + \cos\theta\hat{j}$ and $\vec{c} = \hat{k}$, then angle between $\vec{r}$ and $\vec{a}$ is

(a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1}{3}\right)$
(c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{2}$

8. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then one of the roots of the equation $x^2 + (2c + a)x + c^2 + ac + b = 0$ is

(a) c (b) $d - c$
(c) $2c$ (d) $2d$

9. Let f be a differentiable function satisfying the condition $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y\ (\neq 0) \in R$ and $f(y) \neq 0$. If $f'(1) = 2$, then $f'(x)$ is equal to

(a) $2f(x)$ (b) $\frac{f(x)}{x}$
(c) $2xf(x)$ (d) $\frac{2f(x)}{x}$

10. The number of solutions of the equation $\cos^{-1} x + \cos^{-1}\sqrt{1-x^2} = \pi$ is

(a) 1 (b) 2
(c) 0 (d) none of these

SECTION-II

Numerical Answer Type

11. If $(1+x)^n = \sum_{r=1}^{n} C_r x^r$, then the value of $C_1 + 2C_2 + 3C_3 + ... + nC_n$ equals $n \cdot k^{n-1}$. Find k

12. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals ________.

13. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then $n =$ ________.

14. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ________.

15. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The number of distinct complex numbers z satisfying
$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$ is equal to ________.

SOLUTIONS

1. (b) : Let tangent at $P(at^2, 2at)$ makes an angle θ with x-axis, then $\tan\theta = \frac{1}{t}$

Projection of BC on tangent $= BC\sin\theta$

$= \frac{4a}{\sqrt{1+t^2}} \geq 2a\sqrt{2}$ (as $-1 \leq t \leq 1$).

2. (b) : Given expression will have least value if $2bx - [x^2 + b^2 + \sin^2 x]$ is maximum

$x^2 + b^2 + \sin^2 x - 2bx$ is minimum

$(x - b)^2 + \sin^2 x$ is minimum

Now $|x - b|$ and $|\sin x|$ are minimum if $x = 0$, $b = 2$

So, least value is $-\frac{1}{4}$

3. (a) : Since, $4 \sin A \cos B = 1$, so A and B can not be $\frac{\pi}{2}$

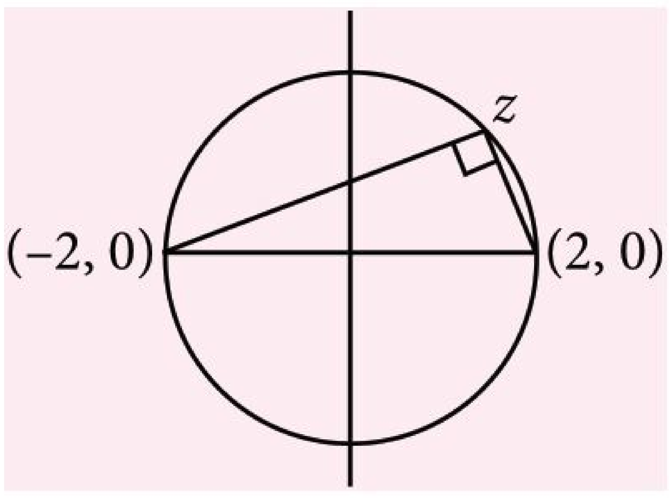
[As if $B = \frac{\pi}{2}$, then $\cos B = 0$ and if $A = \frac{\pi}{2}$, $\tan A$ is not defined]

$C = \frac{\pi}{2}, B = \frac{\pi}{2} - A \Rightarrow 4 \sin A \cos\left(\frac{\pi}{2} - A\right) = 1$

$\Rightarrow \sin^2 A = \frac{1}{4} \Rightarrow \sin A = \frac{1}{2} \Rightarrow A = \frac{\pi}{6} \Rightarrow B = \frac{\pi}{3}$

So angles are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ which are in A.P.

4. (c) : Clearly, $\text{Arg}\left(\frac{z-2}{z+2}\right) = \pm\frac{\pi}{2}$



$\Rightarrow \text{Arg}\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \pm\frac{\pi}{2}$

So z_1, z_2, z_3 will be the vertices of a right angled triangle.

5. (c) : $(a - b)^2 - c^2 = 0$

$\Rightarrow (a - b - c)(a - b + c) = 0$

If $a - b = c \Rightarrow ax + by + a - b = 0$

$\Rightarrow (x + 1)a + b(y - 1) = 0 \Rightarrow x = -1, y = 1$

If $-a + b = c \Rightarrow ax + by + b - a = 0$

$\Rightarrow (x - 1)a + (y + 1)b = 0$

$\Rightarrow (x-1) + (y+1)\frac{b}{a} = 0 \Rightarrow x = 1, y = -1$

Equation of line passing through both points $(-1, 1)$ and $(1, -1)$ is $y = -x$.

6. (d) : $g'(x) = f'((\tan x - 1)^2 + 3) = (2\tan x - 2)\sec^2 x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is increasing

So $f'((\tan x - 1)^2 + 3) > f'(3) = 0$

$\forall x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Also $(\tan x - 1) > 0 \ \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

So, $g(x)$ is increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

7. (c) : Since $\vec{a} \cdot \vec{b} = 0 = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular.

Now $\vec{r} = t(\vec{a} + \vec{b} + \vec{c}) = t((\cos\theta - \sin\theta)i$
$+ (\cos\theta + \sin\theta)\hat{j} + \hat{k})$

Let angle between $\vec{r}$ and $\vec{a}$ be α, then

$$\cos\alpha = \frac{\vec{r} \cdot \vec{a}}{|\vec{r}||\vec{a}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

8. (b) : Let $f(x) = x^2 + ax + b$, then
$x^2 + (2c + a)x + c^2 + ac + b = f(x + c)$
If c, d are the roots of $f(x)$
$\therefore$ $c^2 + ac + b = 0$ and $c + d = -a$ ...(i)
Let α, β are the roots of $f(x + c) = 0$
$\therefore$ Product of roots, $\alpha\beta = c^2 + ac + b = 0$
$\therefore$ One root is 0. ...(ii)
Also, sum of roots $= -(2c + a)$
$\Rightarrow$ $\alpha + \beta = -(2c + a)$
$\Rightarrow$ $0 + \beta = -(2c - c - d) \Rightarrow \beta = d - c$
Thus roots of $f(x + c) = 0$ will be 0, $(d - c)$.

9. (d) : We have

$$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$$

Replacing x and y both by 1,

$$f(1)=\frac{f(1)}{f(1)}=1$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)\lim_{h\to 0}\left\{\frac{\frac{f(x+h)}{f(x)}-1}{h}\right\}=f(x)\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)-1}{h}$$

$$=\frac{f(x)}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{h/x}$$

$$=\frac{f(x)}{x}f'(1)=\frac{2f(x)}{x} \qquad (\because\ f'(1)=2)$$

10. (a) : $\cos^{-1}\sqrt{1-x^2}=\pi-\cos^{-1}x=\cos^{-1}(-x)$

$\Rightarrow \sqrt{1-x^2}=-x \Rightarrow x<0$

Squaring, $1 - x^2 = x^2 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

Since $x < 0$ $\therefore$ $x=-\frac{1}{\sqrt{2}}$ is the only solution.

11. (2) : $C_1 + 2C_2 + 3C_3 + ... + nC_n$

$$=n+2.\frac{n(n-1)}{2!}+\frac{3n(n-1)(n-2)}{3!}+.....+n$$

$$=n\left[1+(n-1)+\frac{(n-1)(n-2)}{2!}+.....+1\right]$$

$= n(1 + 1)^{n-1} = n2^{n-1} \Rightarrow k = 2$

12. (4) : $F(x)=\int_0^x f(t)dt \Rightarrow F(x^2)=\int_0^{x^2} f(t)dt=x^2(1+x)$

$\Rightarrow f(x^2)\cdot 2x = 2x + 3x^2 \Rightarrow f(4) = 4$

13. (6) : Let T_r, T_{r+1}, T_{r+2} be three consecutive terms.
Coefficients of

$T_r = {}^{n+5}C_{r-1}, T_{r+1} = {}^{n+5}C_r, T_{r+2} = {}^{n+5}C_{r+1}$

Now, $\frac{T_{r+1}}{T_r}=\frac{10}{5}$

$$\Rightarrow \frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}}=2\Rightarrow\frac{n+6-r}{r}=2$$

$\Rightarrow n + 6 - r = 2r \Rightarrow 3r = n + 6$ ...(i)

Also, $\frac{T_{r+2}}{T_{r+1}}=\frac{14}{10}\Rightarrow\frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r}=\frac{7}{5}$

$$\Rightarrow \frac{n+5-r}{r+1}=\frac{7}{5}$$

$$\Rightarrow \frac{n+6}{r+1}=\frac{12}{5}\Rightarrow\frac{3r}{r+1}=\frac{12}{5} \quad \text{[Using (i)]}$$

$\Rightarrow 15r = 12r + 12 \Rightarrow 3r = 12 \Rightarrow r = 4$
$\therefore$ $n = 3(4) - 6 = 6$

14. (2) : $f(x) = x^3 + e^{x/2} \Rightarrow f'(x)=3x^2+\frac{e^{x/2}}{2}$

we have, $g(f(x)) = x$
On differentiating both sides, w.r.t. 'x', we get
$g'(f(x))\, f'(x) = 1$
Putting $x = 0$ in above equation, we get
$g'(f(0))\, f'(0) = 1$
But $f(0) = 1$ and $f'(0)=\frac{1}{2}$

$\therefore$ $g'(1)=\frac{1}{f'(0)}=\frac{1}{1/2}=2$

15. (1) : $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0\Rightarrow\begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$

$\Rightarrow z\{(z + \omega^2)(z + \omega) - 1 - \omega(z + \omega - 1)$
$+ \omega^2(1 - z - \omega^2)\} = 0$
$\Rightarrow z^3 = 0$
$\Rightarrow z = 0$ is the only solution.

MONTHLY TEST DRIVE CLASS XI ANSWER KEY

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(c)	**5.**	(a)
6.	(c)	**7.**	(b,c)	**8.**	(b,c)	**9.**	(a,c)	**10.**	(c,d)
11.	(b,c)	**12.**	(a,b)	**13.**	(a,b,c,d)			**14.**	(c)
15.	(b)	**16.**	(b)	**17.**	(785)	**18.**	(7)	**19.**	(6)
20.	(3)								

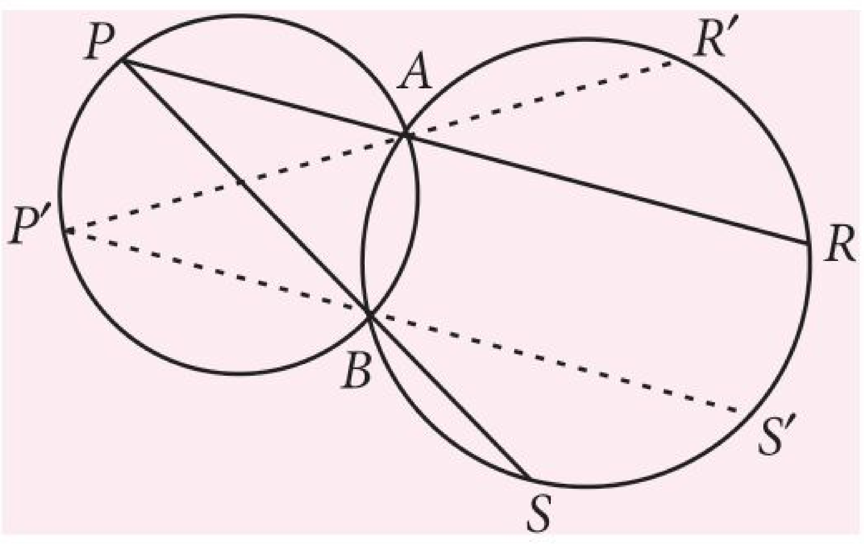
OLYMPIAD CORNER

1. We are considering triangles ABC in space.

(a) What conditions must be fulfilled by the angles α, β, γ of triangle ABC in order that there exists a point P in space such that $\angle APB, \angle BPC, \angle CPA$ are right angles?

(b) Let d be the maximum distance among PA, PB, PC and let h be the longest altitude of triangle ABC. Show that $(\sqrt{6}/3)h \le d \le h$.

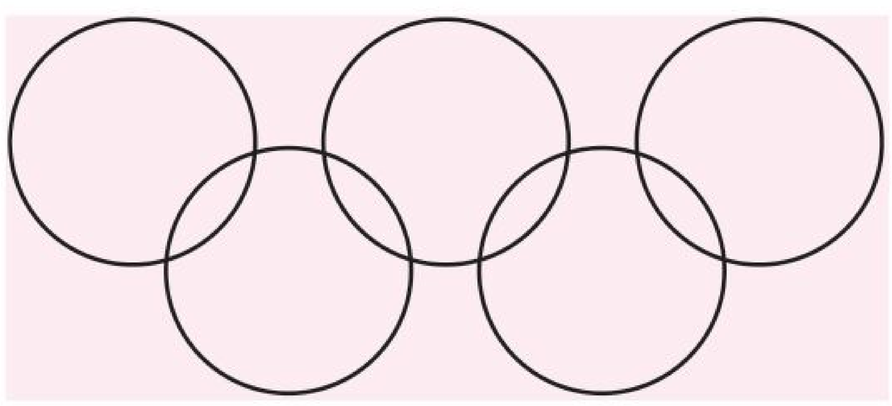
2. Two circles intersect at A and B. P is any point on an arc AB of one circle. The lines PA, PB intersect the other circle at R and S, as shown below. If P' is any other point on the same arc of the first circle and if R', S' are the points in which the lines $P'A$, $P'B$ intersect the other circle, prove that the arcs RS and $R'S'$ are equal.



3. For any positive integer n, evaluate a_n/b_n, where

$$a_n = \sum_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}, b_n = \prod_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}.$$

4. There are 9 regions inside the 5 rings of the Olympics. Put a different whole number from 1 to 9 in each so that the sum of the numbers in each ring is the same. What are the largest and the smallest values of this common sum?



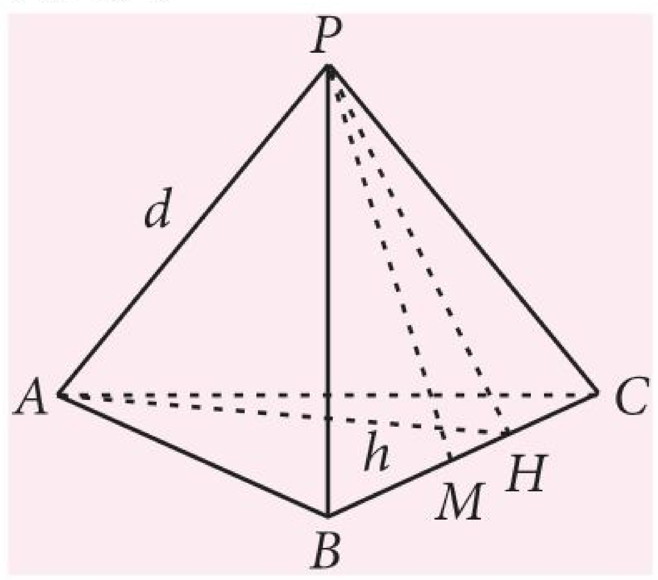
5. If m and n are positive integers such that $\frac{m+n}{m^2+mn+n^2} = \frac{4}{49}$, then find $m + n$.

6. An equilateral triangle of side length 2 units is inscribed in a circle. Find the length of a chord of this circle which passes through the midpoints of two sides of this triangle.

7. In a soccer tournament eight teams play each other once, with two points awarded for a win, one point for a draw and zero for a loss. How many points must a team score to ensure that it is in the top four (*i.e.* has more points than at least four other teams)?

SOLUTIONS

1. (a) By Pythagoras theorem, we have
$AB^2 = PA^2 + PB^2$, $AC^2 = PA^2 + PC^2$
and $BC^2 = PB^2 + PC^2$.



Hence $AB^2 + AC^2 - BC^2 = 2PA^2 > 0$, thus $\angle BAC < 90°$, *i.e.*, $\alpha < 90°$.
Similarly, we get $\beta < 90°$ and $\gamma < 90°$. Conversely, if α, β and γ are all acute angles, we may prove that there exists a point P such that $\angle APB, \angle ABC, \angle CPA$ are right angles.

(b) We may without loss of generality assume that $PA \ge PB \ge PC$, so $PA = d$, Because $AB^2 = AP^2 + BP^2 \ge AP^2 + CP^2 = AC^2$ and $AC^2 = AP^2 + CP^2 \ge BP^2 + CP^2 = BC^2$, we have $AB \ge AC \ge BC$.
Let H be the foot of the perpendicular from A to BC, then AH is the longest altitude of ΔABC, so $AH = h$. As

$AP \perp BP$ and $AP \perp CP$, AP is perpendicular to the plane of BPC. Thus $AP \perp BC$ and $AP \perp PH$ so that $AP < AH$, *i.e.*, $d < h$. ...(1)

Because $AP \perp BP$ and $AH \perp BC$, we get BC is perpendicular to the plane of APH. Thus, we have $BC \perp PH$.

Let M be the midpoint of BC, then $PH \le PM$. As $\angle BPC = 90°$, we have $PM = BM = MC = \frac{1}{2}BC$. Hence, $2PH \le 2PM = BC$, so that

$$4PH^2 \le BC^2 = PB^2 + PC^2 \le 2PA^2 \quad ...(2)$$

As $\angle APH = 90°$, we get

$$PH^2 = AH^2 - AP^2 = h^2 - d^2 \quad ...(3)$$

From (2) and (3), we have

$$4(h^2 - d^2) \le 2d^2,\ i.e.,\ 2h^2 \le 3d^2,$$

from which, we have

$$\frac{\sqrt{6}}{3}h \le d. \quad ...(4)$$

From (1) and (4) we obtain $\frac{\sqrt{6}}{3}h \le d < h$, as required.

2. Because opposite angles of a cyclic quadrilateral are supplementary, we have that $\angle PBA = \pi - \angle ABS = \angle ARS$. Similarly $\angle PAB = \angle BSR$. Thus ΔPAB and ΔPSR are similar, from which

$$\frac{PA}{PS} = \frac{PB}{PR} = \frac{AB}{RS}$$

(Notice that this also gives the 'power of the point' result for P, $PA \cdot PR = PB \cdot PS$)

Similarly $\frac{P'A}{P'S} = \frac{P'B}{P'R'} = \frac{AB}{R'S'}$

Consider now traingles APS and $AP'S'$. We have $\angle APS = \angle APB = \angle AP'B = \angle AP'S'$, because P, P' lie on the same arc of chord AB of the one circle. From the fact that S and S' lie on the same arc of chord AB of the second circle $\angle AS'P' = \angle ASP$. But then ΔAPS and $\Delta AP'S'$ are similar. Thus

$$\frac{PA}{PS} = \frac{P'A}{P'S'}. \text{ So } \frac{AB}{RS} = \frac{AB}{R'S'}$$

Thus $RS = R'S'$ and the arcs are equal.

3. Using De Moivre's theorem

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n,$$

one finds easily that

$$\sin n\theta = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta$$

So $\sin(2n+1)\theta = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \cos^{2n-2k}\theta \sin^{2k+1}\theta$

$$= \tan\theta \cos^{2n+1}\theta \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta.$$

Thus $\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta = 0$

For $\theta = \frac{j\pi}{2n+1}, 1 \le j \le n$

So $\tan^2\frac{j\pi}{2n+1}, 1 \le j \le n,$ are the roots of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} x^k = 0$ and thus also of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^{n-k} = 0.$ Since a_n and b_n are the sum and product of the roots, respectively, we have

$$a_n = \binom{2n+1}{2} = n(2n+1) \text{ and } b_n = \binom{2n+1}{2n} = 2n+1,$$

and so $\frac{a_n}{b_n} = n$

4. For the five rings, we have

$a + b = b + c + d = d + e + f = f + g + h$

$= h + i = N.$ ...(1)

Since we are dealing with the nine non-zero decimal digits, we have $\sum^{9} j = 9(10)/2 = 45.$ The five regions sum to a common[1] N for 45/5 = 9 but then one pair must be 9 + 0 or one triplet 9 + 0 + 0, which isn't allowed. So $N > 9$. Since $a + b = h + i$, there must be at least two pairs of decimal digits that sum to N. For $10 \le N \le 15$, we have

$$N = 9 + a = 8 + (1 + a) = ..., \text{ for } 1 \le a \le 6$$

while $N = 16 = 9 + 17$ and $N = 17 = 9 + 8$ only. So $N \leq 15$.
From (1), $a + b = b + c + d$ or $a = c + d$ ...(2)
and $h + i = f + g + h$ or $i = f + g$ ...(3)
The five central digits must equal $45 - 2N$

$$(c + d) + e + (f + g) = a + e + i = 45 - 2N$$

So, we have

N	$2N$	$45 - 2N$	a, e, i
10	20	25	9, 8 – no digit available
11	22	23	9, 8, 6;...
12	24	21	9, 8, 4;...
13	26	19	9, 8, 2;...
14	28	17	9, 7, 1;...
15	30	15	9, 5, 1;...

So $11 \leq N \leq 15$.

5. Assume that $m + n = 4k$, $m^2 + mn + n^2 = 49k$. Then $m^2 + 2mn + n^2 = (m + n)^2 = (4k)^2 = 16k^2$,
hence $mn = 16k^2 - 49k$. Since $mn > 0$, from $16k^2 - 49k > 0$ we find that $k > 3$. Since we also have the identity $mn = \left(\frac{m+n}{2}\right)^2 - \left(\frac{m-n}{2}\right)^2$,

where $\left(\frac{m-n}{2}\right)^2 \geq 0$,

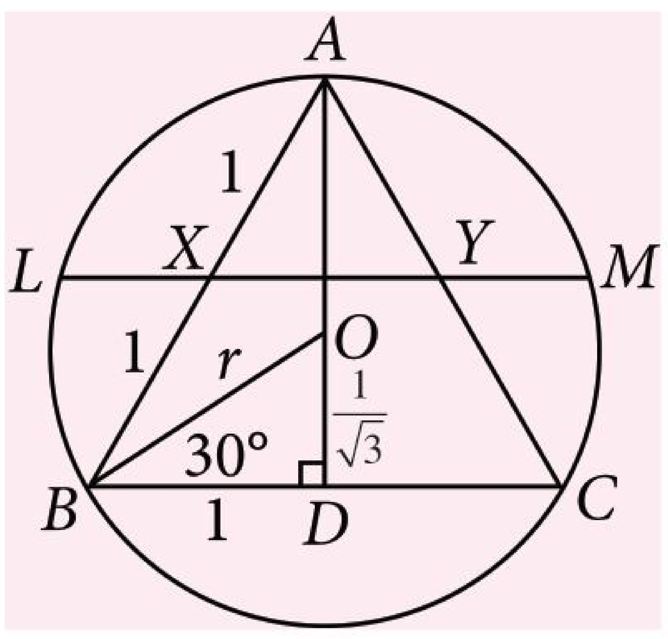
we also find that

$$0 \leq \left(\frac{m+n}{2}\right)^2 - mn = \left(\frac{4k}{2}\right)^2 - \left(16k^2 - 49k\right)$$

$= 49k - 12k^2$, from which $k \leq 4$ follows.
Hence $k = 4$ (since it must be an integer), and from $m + n = 16$ and $mn = 6$, $n = 10$, and $m^2 + mn + n^2 = 196$ follow. Indeed,

$$\frac{m+n}{m^2+mn+n^2} = \frac{16}{196} = \frac{4}{49}.$$

6.



Let the circumcircle of the equilateral ΔABC have centre O (0, 0) and radius r, Join A to D, the mid-point of BC, then AD passes through O and is perpendicular to BC. Draw OB. Let the chord LM cut the sides AB and AC of ΔABC at X and Y. Then $LXYM$ is parallel to BC.

$OD = \tan 30° = \frac{1}{\sqrt{3}}$

Then $D = \left(0, -\frac{1}{\sqrt{3}}\right)$,

$A = \left(0, \frac{2}{\sqrt{3}}\right)$ and

$B = \left(-1, -\frac{1}{\sqrt{3}}\right)$ and the equation of the circle is $x^2 + y^2 = \frac{4}{3}$.

Now X is the mid-point of AB so

$$X = \left(-\frac{1}{2}, \frac{1}{2}\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right) = \left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$$

To find the x – coordinates of L and M, substitute the y - coordinate of X in the equation $x2 + y2 = \frac{4}{3}$, *i.e.*

$$x^2 + \left(\frac{1}{2\sqrt{3}}\right)^2 = \frac{4}{3}$$

$$\Rightarrow \ x^2 = \frac{4}{3} - \frac{1}{12} = \frac{5}{4}$$

$$\Rightarrow \ x = \pm\frac{\sqrt{5}}{2}$$

Thus the x-coordinate of L is $-\frac{\sqrt{5}}{2}$ and the x-coordinate of M is $\frac{\sqrt{5}}{2}$, so the length of LM is $\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2} = \sqrt{5}$.

7. Since there are 8 teams, there are 7 rounds of four matches and thus a total of $7 \times 8 = 56$ points available.
Consider a team with 10 points. It is possible to have 5 teams on 10 points and 3 teams on 2 points when each of the top 5 draws with each other, each of the bottom 3 draws with each other and each of the top 5 wins against each of the bottom 3. So, 10 points does not guarantee a place in the top 4.
Consider a team with 11 points. If this team was fifth, then the number of points gained by the top 5 teams is ≥ 55. This is impossible as the number of points shared by the bottom 3 teams is then 1, as these 3 teams must have at least $3 \times 2 = 6$ points between them for the games played between themselves. Hence 11 points is sufficient to ensure a place in the top 4. Thus 11 points are required.

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

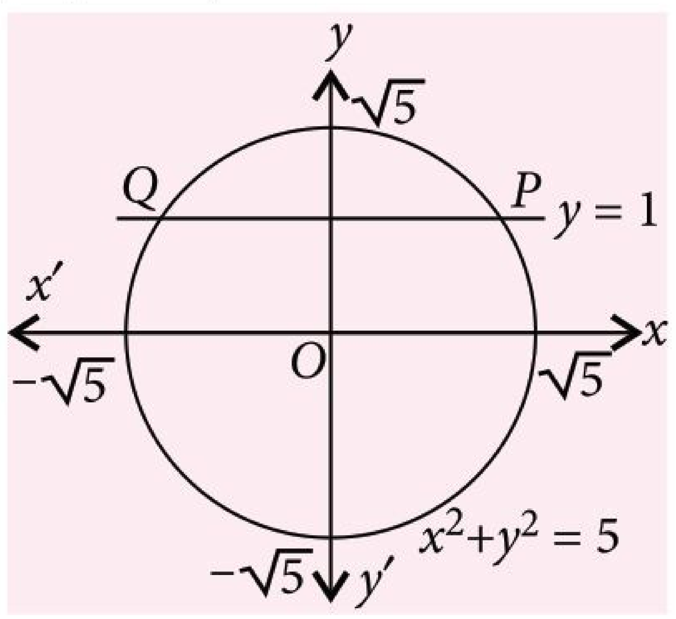
1. Find the angle of intersection of curves, $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$, where $[\cdot]$ denotes the greatest integer function.

(Neeraj, Delhi)

Ans. We know that, $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

$\Rightarrow \quad y = [|\sin x| + |\cos x|] = 1$

Let P and Q be the points of intersection of given curves. Clearly, the given curves meet at points where $y = 1$, so we get $x^2 + 1 = 5$



$\Rightarrow \quad x = \pm 2$

$\therefore \quad P(2, 1)$ and $Q(-2, 1)$

On differentiating $x^2 + y^2 = 5$ w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(2,1)} = -2 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,1)} = 2$$

Clearly, the slope of line $y = 1$ is zero and the slope of the tangents at P and Q are (-2) and (2), respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

2. Evaluate : $\displaystyle\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

(Varnika, U.P.)

Ans. Let

$$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx = \int \frac{\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}\, dx$$

$$= \int \left(1 - \frac{4}{\pi}\cos^{-1}\sqrt{x}\right) dx = x - \frac{4}{\pi}\int 1 \cdot \cos^{-1}\sqrt{x}\, dx$$

$$= x - \frac{4}{\pi}\left[\cos^{-1}\sqrt{x} \cdot x - \int x \cdot \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}\, dx\right]$$

$$\therefore \quad I = x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx$$

Put $x = \cos^2\theta$; then $dx = -2\cos\theta \cdot \sin\theta\, d\theta$

$$\therefore \quad \int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx = \int \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}(-2\cos\theta \cdot \sin\theta)\, d\theta$$

$$= -\int 2\cos^2\theta\, d\theta = -\int (1 + \cos 2\theta)\, d\theta = -\left\{\theta + \frac{\sin 2\theta}{2}\right\} + c$$

$$= -\cos^{-1}\sqrt{x} - \sqrt{x}.\sqrt{1-x} + c$$

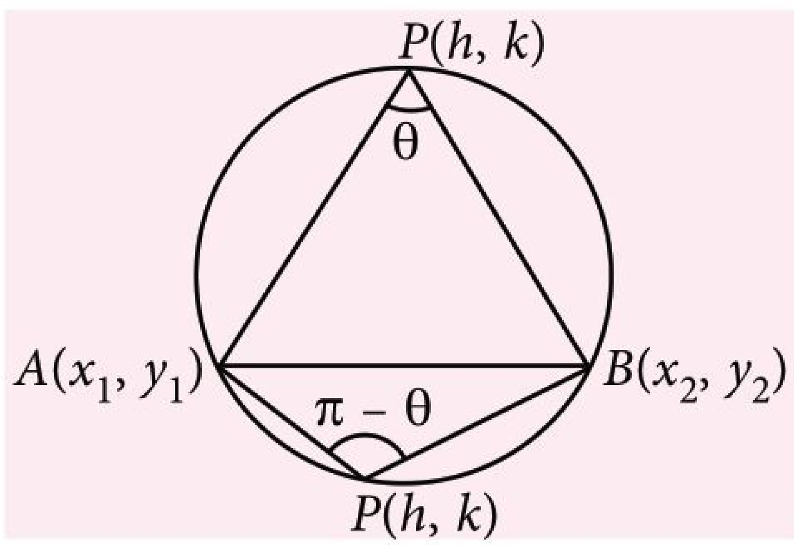
$$\therefore \; I = x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left\{-\cos^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + c\right\}$$

$$= x + \frac{2}{\pi}(1 - 2x)\cos^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x(1-x)} + c_1, \text{ where } c_1 = \frac{-2}{\pi}c$$

3. Find the general equation of the circle passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$.

(Tanvi Sharma, Kerala)

Ans. Let $P(h, k)$ be any point on the circle passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$. Since the angle in the same segment of a circle is always same.



Therefore, $\angle APB = \theta$ or $\pi - \theta$, where θ is some angle.

Now, m_1 = Slope of $AP = \dfrac{k - y_1}{h - x_1}$,

and, m_2 = Slope of $BP = \dfrac{k - y_2}{h - x_2}$

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow \tan\theta = \pm\frac{\frac{k - y_1}{h - x_1} - \frac{k - y_2}{h - x_2}}{1 + \frac{k - y_1}{h - x_1} \times \frac{k - y_2}{h - x_2}}$$

$$\Rightarrow \quad \tan\theta = \pm\frac{(h - x_2)(k - y_1) - (h - x_1)(k - y_2)}{(h - x_1)(h - x_2) + (k - y_1)(k - y_2)}$$

$\Rightarrow \quad (h - x_1)(h - x_2) + (k - y_1)(k - y_2)$
$= \pm\cot\theta\{(h - x_2)(k - y_1) - (h - x_1)(k - y_2)\}$

Hence, the locus of (h, k) is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$
$= \pm\cot\theta\{(x - x_2)(y - y_1) - (x - x_1)(y - y_2)\}$

$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$
$\pm\cot\theta\{x(y_1 - y_2) + y(x_2 - x_1) + x_1 y_2 - x_2 y_1\} = 0$

$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot\theta \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

CONCEPT MAP

SEQUENCES AND SERIES

Class XI

SERIES

If $a_1, a_2, ..., a_n$ is a sequence, then the expansion $a_1 + a_2 + ... + a_n + ...$ is called the series.

SEQUENCE

A sequence is a function from natural number N (domain) to real numbers (codomain)

Progression

If the terms of a sequence are written under specific conditions, then the sequence is called progression.

Types

Arithmetic Progression (A.P.)

A sequence whose terms increases or decreases by a fixed number.

- n^{th} **term :** $T_n = a + (n - 1)d$, where d = common difference $= T_n - T_{n-1}$, a = first term,
- n^{th} **term from end :** $T_n' = l - (n - 1)d$, where l = last term
- **Sum of n terms :** $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$

Geometric Progression (G.P.)

A sequence of non-zero numbers for which the ratio of a term to its just preceding term is always constant.

- n^{th} **term :** $T_n = ar^{n-1}$, where r (common ratio) $= T_n/T_{n-1}$, a = first term.
- n^{th} **term from end :** $T_n' = l/r^{n-1}$, l = last term
- **Sum of n terms :** $S_n = \begin{cases} \frac{a(r^n-1)}{r-1}, r>1 \\ \frac{a(1-r^n)}{1-r}, r<1 \\ an, \quad r=1 \end{cases}$; $S_\infty = \frac{a}{1-r}$, if $|r|<1$.

Note : If $|r| \geq 1$, S_∞ does not exist.

Harmonic Progression (H.P.)

A sequence $a_1, a_2, a_3, ...$ in which reciprocal of terms form an A.P.

- n^{th}**term:** $T_n = \frac{1}{\frac{1}{a_1}+(n-1)\left(\frac{1}{a_2}-\frac{1}{a_1}\right)} = \frac{a_1a_2}{a_2+(n-1)(a_1-a_2)}$

Note : No term of an H.P. can be zero.

Sum of n terms of special series

- Sum of n natural numbers, $\sum n = \frac{n(n+1)}{2}$
- Sum of squares of n natural numbers, $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
- Sum of cubes of n natural numbers, $\sum n^3 = \frac{n^2(n+1)^2}{4} = \left(\sum n\right)^2$

Basic Properties

- If a constant is added / subtracted / multiplied / divided to each term of an A.P., then the resulting sequence is also an A.P.
- Selection of terms in an A.P.
  - Any three numbers in A.P. can be taken as $a - d, a, a + d$.
  - Any four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$.
- If each term of a G.P. is multiplied / divided by a same non-zero number, then the resulting sequence is also in G.P.
- Selection of terms in a G.P.
  - Any three numbers in G.P. can be taken as $a/r, a, ar$.
  - Any four numbers in G.P. can be taken as $a/r^3, a/r, ar, ar^3$.

Arithmetic Mean (A.M.)

- For two numbers a and b, A.M. is $\frac{a+b}{2}$.
- $A_k = a + k\left(\frac{b-a}{n+1}\right), \forall k = 1, 2, ..., n$

Where $A_1, A_2, ..., A_n$ are n arithmetic means inserted between two numbers a and b.

Geometric Mean (G.M.)

- For two numbers a and b, G.M. is $\sqrt{ab}$.
- $G_k = a\left(\frac{b}{a}\right)^{k/n+1} \quad \forall k = 1, 2, 3, ..., n$

Where $G_1, G_2, ..., G_n$ are n geometric means inserted between two numbers a and b.

Harmonic Mean (H.M.)

- For two numbers a and b, H.M. is $2ab/(a + b)$.
- $H_n = \frac{(n+1)ab}{an+b} \;\forall\; n = 1, 2, ...$

Where a and b are two numbers and $H_1, H_2, ..., H_n$ are n harmonic means inserted between them.

DEFINITE INTEGRALS AND APPLICATION OF INTEGRALS

Class XII

CONCEPT MAP

Fundamental Theorems of Calculus

- **First Fundamental Theorem :** Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.
- **Second Fundamental Theorem :** Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and $F(x)$ be an integral of $f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Solving by Substitution

When definite integral is to be found by substitution, change the lower and upper limits of integration. If substitution is $t = f(x)$ and lower limit of integration is a and upper limit is b, then new lower and upper limits will be $f(a)$ and $f(b)$ respectively.

For any two values a and b, we have $\int_a^b f(x)\,dx = [F(x) + c]_a^b = F(b) - F(a)$

Here, $F(x)$ is anti derivative of function $f(x)$.

Limit of Sum

$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + ... + f(a + (n-1)h)]$,

where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

DEFINITE INTEGRALS

Properties

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

  In particular $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,

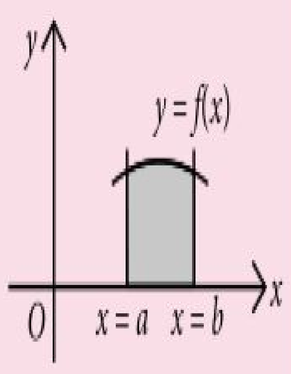
  where $a < c < b$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_{-a}^a f(x)dx = \begin{cases} 0 & \text{, if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{, if } f(-x) = f(x) \end{cases}$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

APPLICATION OF INTEGRALS

Area Under Simple Curves

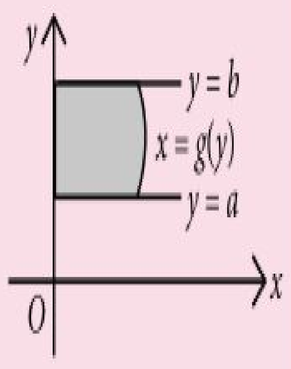
- Area $= \int_a^b y\,dx$

  $= \int_a^b f(x)dx$ (where $b > a$)

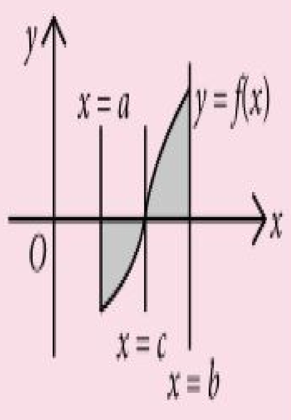


- Area $= \int_a^b x\,dy$

  $= \int_a^b g(y)dy$ (where $b > a$)



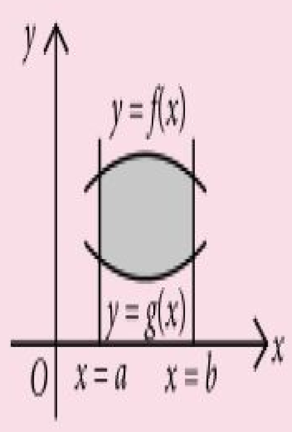
- Area $= \left|\int_a^c f(x)dx\right| + \int_c^b f(x)dx$



Area Between Two Curves

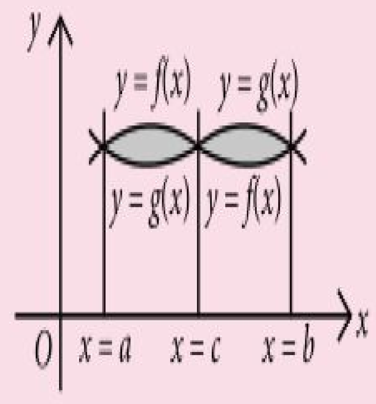
- Area $= \int_a^b [f(x) - g(x)]dx$,

  $f(x) \geq g(x)$ in $[a, b]$

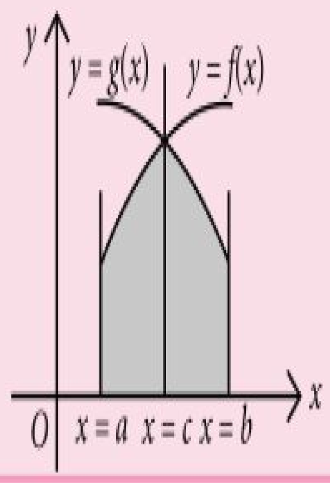


- Area $= \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$

  where $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$



- Area $= \int_a^c f(x)dx + \int_c^b g(x)dx$



JEE WORK OUTS

One or More Than One Option(s) Correct Type

1. Let A, B and C be three angles such that $A = \frac{\pi}{4}$ and $\tan B \tan C = p$. The set of all possible values of p such that A, B, C are the angles of a triangle contains
   (a) $(-\infty, 0)$ (b) $(0, 1)$
   (c) $(1, 3+2\sqrt{2})$ (d) $[3+2\sqrt{2}, \infty)$

2. Let ABC be a triangle with $\angle BAC = 120°$ and $AB \cdot AC = 1$. Also, let AD be the length of the angle bisector of $\angle A$ of the triangle. Then
   (a) Minimum value of AD is $\frac{1}{2}$
   (b) Maximum value of AD is $\frac{1}{2}$
   (c) AD is minimum when ΔABC is isosceles
   (d) AD is maximum when ΔABC is isosceles

3. 8 players $P_1, P_2, \ldots, P_8$ of equal strength play in a knockout tournament. Assuming that players in each round are paired randomly, the probability that the player P_1 loses to the eventual winner is
   (a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{7}{8}$

4. If z_1 and z_2 are two complex numbers such that $|z_1| = 2$ and $(1-i)z_2 + (1+i)\bar{z}_2 = 8\sqrt{2}$, then the minimum value of $|z_1 - z_2|$ is
   (a) 1 (b) 2 (c) 3 (d) 4

5. Let L_1 and L_2 be the lines $\vec{r} = (2\hat{i}+\hat{j}-\hat{k}) + \lambda(\hat{i}+2\hat{k})$ and $\vec{r} = (3\hat{i}+\hat{j}) + \mu(\hat{i}+\hat{j}-\hat{k})$. If the plane π which contains L_1 and parallel to L_2 meets the coordinate axes at A, B and C respectively, then the volume of the tetrahedron $OABC$ is
   (a) $\frac{4}{9}$ (b) $\frac{4}{3}$ (c) $\frac{2}{9}$ (d) $\frac{2}{3}$

6. Which of the following statement(s) is/are correct?
   (a) Rolle's theorem is applicable to the function $F(x) = 1 - \sqrt[5]{x^6}$ on the interval $[-1, 1]$.
   (b) The domain of definition of the function $F(x) = \dfrac{\log_4(6-[x]-[x]^2)}{x^2+x-2}$ is $(-3, -2) \cup (-2, 1) \cup (1, 2)$.
   (where $[x]$ denotes the largest integer less than or equal to x)
   (c) The value of a for which the function $F(\theta) = a \sin\theta + \frac{1}{3}\sin 3\theta$ has an extremum at $\theta = \pi/3$ is -2.
   (d) The value of $\sum_{k=1}^{2010} \dfrac{\{x+k\}}{2010}$ is $\{x\}$.
   (where $\{x\}$ denotes the fractional part of x).

7. Let a, b, c be distinct complex numbers with $|a| = |b| = |c| = 1$ and z_1, z_2 be the roots of the equation $az^2 + bz + c = 0$ with $|z| = 1$. Let P and Q represent the complex numbers z_1 and z_2 in the argand plane with $\angle POQ = \theta$, $0° < \theta < 180°$ (where O being the origin), then
   (a) $b^2 = ac;\ \theta = \frac{2\pi}{3}$ (b) $\theta = \frac{2\pi}{3};\ PQ = \sqrt{3}$
   (c) $PQ = 2\sqrt{3};\ b^2 = ac$ (d) $\theta = \frac{\pi}{3};\ b^2 = ac$

8. Which of the following statement (s) is/are true?
   (a) Maximum value of P such that 3^P divides 100! is 48.
   (b) Maximum value of P such that 3^P divides 50! is 22.

(c) Maximum value of P such that 3^P divides 25! is 10.

(d) none of these

9. In a ΔABC, if $A = (1, 2)$ and internal angle bisectors through B and C are $y = x$ and $y = -2x$. The inradius of ΔABC is equal to

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{2}}$

10. If $I_n = \int_0^1 x^n \tan^{-1} x\,dx$ and $a_n I_{n+2} + b_n I_n = c_n$ $\forall\ n \in N, n \ge 1$, then

(a) $a_1, a_2, a_3, ...$ are in A.P.

(b) $b_1, b_2, b_3, ...$ are in A.P.

(c) $c_1, c_2, c_3, ...$ are in G.P.

(d) $a_1, a_2, a_3, ...$ are in H.P.

11. The equation of a circle is $S_1 \equiv x^2 + y^2 = 1$. The orthogonal tangents to S_1 meet at another circle S_2 and the orthogonal tangents to S_2 meet at the third circle S_3. Then

(a) radius of S_2 and S_3 are in the ratio $1 : \sqrt{2}$

(b) radius of S_2 and S_3 are in the ratio $1 : 2$

(c) the circles S_1, S_2 and S_3 are concentric

(d) none of these

12. Tangent is drawn at any point (x_1, y_1) other than the vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2), then

(a) x_1, a, x_2 are in G.P.

(b) $\frac{y_1}{2}, a, y_2$ are in G.P.

(c) $-4, \frac{y_1}{y_2}, \frac{x_1}{x_2}$ are in G.P.

(d) $x_1x_2 + y_1y_2 = a^2$

13. If $\log_2(5\cdot 2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in A.P., then x is equal to

(a) $\frac{\log 5}{\log 2}$ (b) $\log_2 (0.4)$

(c) $1 - \frac{\log 5}{\log 2}$ (d) $\frac{\log 2}{\log 5}$

14. If $\int \operatorname{cosec} 2x dx = f\{g(x)\} + C$, then

(a) range of $g(x) = (-\infty, \infty)$

(b) domain of $f(x) = (-\infty, \infty)-\{0\}$

(c) $g'(x) = \sec^2 x$

(d) $f'(x) = \frac{1}{x}$ for all $x \in (0, \infty)$

15. Six persons stand at random in a queue for buying cinema tickets. Individually three of them have only a fifty rupee note each while each of the other three have a hundred rupee note only. The booking clerk has an empty cash box, probability that six persons get tickets without waiting for change is ...., (cost of one ticket is ₹50 and each person gets one ticket only)

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

16. The mean and variance of seven observations are 8 and 16, respectively. If 5 observations are given by 2, 4, 10, 12, 14, then the product of the remaining two observations is

(a) 45 (b) 48 (c) 40 (d) 49

17. A point on the straight line, $3x + 5y = 15$ which is equidistant from the coordinate axes will lie only in

(a) 1st, 2nd and 4th quadrants

(b) 1st and 2nd quadrants

(c) 4th quadrant

(d) 1st quadrant

18. The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is

(a) 160 (b) 105 (c) 210 (d) 320

19. If $f(x) = \log_e\left(\frac{1-x}{1+x}\right)$, $|x| < 1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $2f(x^2)$ (b) $-2f(x)$ (c) $(f(x))^2$ (d) $2f(x)$

20. If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$, $x \in \left(0, \frac{\pi}{2}\right)$

then $\frac{dy}{dx}$ is equal to

(a) $\frac{\pi}{6} - x$ (b) $2x - \frac{\pi}{3}$ (c) $x - \frac{\pi}{6}$ (d) $\frac{\pi}{3} - x$

Comprehension Type

Paragraph for Q. No. 21 to 23

Let $\vec{r}$ is a position vector of a variable point in cartesian OXY plane such that $\vec{r} \cdot (10\hat{j} - 8\hat{i} - \vec{r}) = 40$ and $p_1 = \max\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$, $p_2 = \min\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$.

A tangent line is drawn to the curve $y = \frac{8}{x^2}$ at the point A with abscissa 2. The drawn line cuts x-axis at a point B.

21. p_2 is equal to
(a) 9 (b) $2\sqrt{2}-1$
(c) $6\sqrt{2}+3$ (d) $9-4\sqrt{2}$

22. $p_1 + p_2$ is equal to
(a) 2 (b) 10 (c) 18 (d) 5

23. $\overrightarrow{AB}\cdot\overrightarrow{OB}$ is
(a) 1 (b) 2 (c) 3 (d) 4

Paragraph for Q. No. 24 to 26

For $n \in N$, we put $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$

24. Value of $2(a_0 + a_1 + ... + a_{n-1}) + a_n$ is
(a) 2^{2n-1} (b) 3^n (c) $3^{n/2}$ (d) $(3^n-1)/2$

25. If $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 = ka_n$, then k equals
(a) 1 (b) 2 (c) 1/2 (d) 0

26. If n is not a multiple of 3, and
$\sum_{r=0}^{n}(-1)^r a_r ({}^nC_r) = k({}^nC_{[n/3]})$, where $[x]$ denotes the greatest integer $\le x$, then k *is* equal to
(a) 1 (b) 0 (c) 3 (d) –1

Paragraph for Q. No. 27 and 28

Consider the set of points (x, y) in the plane which satisfy $x^2 + y^2 \le 100$ and $\sin(x + y) \ge 0$.

27. Let A_1 and A_2 be areas of regions within $x^2 + y^2 \le 100$ which satisfy $\sin(x + y) > 0$ and $\sin(x + y) < 0$, then
(a) $A_1 > A_2$ (b) $A_1 < A_2$
(c) $A_1 = A_2$ (d) $A_1 = 2A_2$

28. The area of the region $x^2 + y^2 \le 100$ and $\sin(x + y) \ge 0$ is
(a) 25π (b) 50π (c) 100π (d) 200π

Matrix Match Type

29. For $0 < \theta < \pi/4$, let $x = \sum_{n=0}^{\infty}(\sin\theta)^{2n}, y = \sum_{n=0}^{\infty}(\cos\theta)^{2n}$,
then match the following columns :

Column-I		Column-II	
(A)	$\sum_{n=0}^{\infty}\sin^{2n}\theta\cos^{2n}\theta$	(P)	$\frac{xy^2}{xy^2-1}$
(B)	$\sum_{n=0}^{\infty}\tan^{2n}\theta$	(Q)	$\frac{y}{y-x}$
(C)	$\sum_{n=0}^{\infty}\sin^{2n}\theta\cos^{4n}\theta$	(R)	$\frac{xy}{xy-1}$
(D)	$\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{4n}\theta$	(S)	$\frac{x^2y}{x^2y-1}$

(a) (A) → (Q) ; (B) → (P) ; (C) → (R) ; (D) → (S)
(b) (A) → (S) ; (B) → (P) ; (C) → (Q) ; (D) → (R)
(c) (A) → (R) ; (B) → (Q) ; (C) → (P) ; (D) → (S)
(d) (A) → (S) ; (B) → (Q) ; (C) → (P) ; (D) → (R)

30. Match the following columns :

Column-I		Column-II	
(A)	If the distance of any point (x, y) from origin is defined as $d(x, y) = 2\,\lvert x\rvert + 3\,\lvert y\rvert$. If perimeter and area of figure bounded by $d(x, y) = 6$ are λ units and μ sq. units respectively, then	(P)	(λ, μ) lies on $x^2 - y^2 = 64$
(B)	If the vertices of a triangle are (6, 0), (0, 6) and (6, 6). If the distance between circumcentre and orthocentre and distance between circumcentre and centroid are λ unit and μ unit respectively, then	(Q)	(λ, μ) lies on $x^2 + y^2 - 6x - 6y = 0$
(C)	The ends of the hypotenuse of a right angled triangle are (6, 0) and (0, 6). If the third vertex is (λ, μ), then	(R)	(λ, μ) lies on $x^2 - 16y = 16$
		(S)	(λ, μ) lies on $x^2 - y^2 = 16$

(a) (A) → (P) ; (B) → (S) ; (C) → (Q)
(b) (A) → (P,R) ; (B) → (R) ; (C) → (P)
(c) (A) → (P) ; (B) → (Q) ; (C) → (R,S)
(d) (A) → (Q) ; (B) → (P,Q) ; (C) → (R)

31. Consider the circles C_1 of radius a and C_2 of radius b, $b > a$ both lying in the first quadrant and touching the coordinate axes. Find the value of b/a if

Column-I		Column-II	
(A)	C_1 and C_2 touch each other	(P)	$2+\sqrt{2}$
(B)	C_1 and C_2 are orthogonal	(Q)	3
(C)	C_1 and C_2 intersect so that the common chord is longest	(R)	$2+\sqrt{3}$
(D)	C_2 passes through the centre of C_1	(S)	$3+2\sqrt{2}$

(a) (A) → (S) ; (B) → (P,S) ; (C) → (Q) ; (D) → (Q)
(b) (A) → (R) ; (B) → (Q) ; (C) → (P,S) ; (D) → (Q)
(c) (A) → (P,Q) ; (B) → (R) ; (C) → (S) ; (D) → (P)
(d) (A) → (S) ; (B) → (R) ; (C) → (Q) ; (D) → (P)

Numerical Answer Type

32. Minimum distance between $y^2 - 4x - 8y + 40 = 0$ and $x^2 - 8x - 4y + 40 = 0$ is $\sqrt{\lambda}$, then value of λ is ______.

33. The value of $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is ______.

34. Number of pairs of positive integers (p, q) whose L.C.M. is 8100, is "K". Then number of ways of expressing K as a product of two co-prime numbers is ______.

35. In ΔABC, if $A - B = 120°$ and $R = 8r$, then the value of $\dfrac{1+\cos C}{1-\cos C}$ equals (All symbols used have their usual meaning in a triangle) ______.

36. In ΔABC, orthocentre is (6,10) and circumcentre is (2, 3) and equation of side BC is $2x + y = 17$. Then the radius of the circumcircle of ΔABC is ______.

37. If $f(x)$ and $g(x)$ are periodic functions with periods 7 and 11 respectively, then the period of $F(x) = f(x)g\left(\dfrac{x}{5}\right) - g(x)f\left(\dfrac{x}{3}\right)$ is ______.

38. How many different nine digit numbers can be formed from the number 22 33 55 888 by rearranging its digits, so that the odd digits occupy even positions?

39. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. As L varies, the absolute minimum value of $OP + OQ$ is (O is origin) ______.

40. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real-valued function defined on the interval $[-10, 10]$ by

$$f(x)=\begin{cases} x-[x], & \text{if } [x] \text{ is odd} \\ 1+[x]-x, & \text{if } [x] \text{ is even}\end{cases}$$

Then the value of $\dfrac{\pi^2}{10}\displaystyle\int_{-10}^{10} f(x)\cos \pi x\, dx$ is ______.

SOLUTIONS

1. **(a, d) :** $\tan B \tan C = p$

$$\Leftrightarrow \frac{\cos(B-C)}{\cos(B+C)}=\frac{p+1}{1-p} \Leftrightarrow \cos(B-C)=\frac{p+1}{\sqrt{2}(p-1)}$$

$$\therefore\ 0 \le (B-C) < \frac{3\pi}{4} \Leftrightarrow \frac{-1}{\sqrt{2}}<\frac{p+1}{\sqrt{2}(p-1)}\le 1$$

$$\Leftrightarrow \frac{p+1}{p-1}+1=\frac{2p}{p-1}>0 \text{ and } \frac{(p+1)-\sqrt{2}(p-1)}{\sqrt{2}(p-1)}\le 0$$

$$\Leftrightarrow p(p-1)>0 \text{ and } [(\sqrt{2}+1)-(\sqrt{2}-1)p](p-1)\le 0$$

$$\Leftrightarrow p\notin[0,1] \text{ and } [(\sqrt{2}+1)^2-p](p-1)\le 0$$

$$\Leftrightarrow p\notin[0,1] \text{ and } p\notin(1, 3+2\sqrt{2})$$

$$\Leftrightarrow p\notin(0, 3+2\sqrt{2}) \Leftrightarrow p<0 \text{ or } p\ge 3+2\sqrt{2}$$

2. **(b, d) :** Let $AB = x$. Then $BC^2 = x^2+\dfrac{1}{x^2}+1$

and $\cos B=\dfrac{2x^2+1}{2\sqrt{x^4+x^2+1}}$ *i.e.*, $\tan B=\dfrac{\sqrt{3}}{2x^2+1}$

Also, $\dfrac{AD}{\sin B}=\dfrac{x}{\sin(B+60°)}$

$$\Rightarrow AD=\frac{x}{\frac{1}{2}+\frac{\sqrt{3}}{2}\cot B}=\frac{x}{x^2+1}$$

$\therefore\ AD=\dfrac{1}{x+\frac{1}{x}}\le\dfrac{1}{2}$, with equality iff $AB = AC = 1$

3. **(b) :** If E_1, E_2, E_3 are the events of P_1 losing to the champion in the 1st, 2nd and 3rd rounds, then the required probability

$$=P(E_1)+P(E_2)+P(E_3)=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

4. **(b) :** $\mathrm{Re}(1-i)z_2=4\sqrt{2} \Rightarrow |(1-i)z_2| \ge 4\sqrt{2}$

$\Rightarrow |z_2| \ge 4 \Rightarrow z_2$ lies either on the circumference or outside the circle with centre at origin and radius 4.

$\therefore\ |z_1 - z_2| \ge 2$

5. **(c) :** Equation of the plane containing L_1 and parallel to L_2 is $\begin{vmatrix} x-2 & y-1 & z+1 \\ 1 & 0 & 2 \\ 1 & 1 & -1\end{vmatrix}=0$

$\Rightarrow 2x - 3y - z - 2 = 0$

This plane meets the axes at $A(1, 0, 0)$, $B(0, -2/3, 0)$ and $C(0, 0, -2)$

$\therefore$ Volume of tetrahedron $OABC = \dfrac{1}{6}|abc|=\dfrac{2}{9}$

6. **(a, d) :** (a) We have, $F(x)=\left(1-x^{6/5}\right)$

Now, $F'(x)=\dfrac{-6}{5}x^{\frac{1}{5}}$ exist $\forall\ x\in(-1,1)$

Also, $F(-1) = 0 = F(1)$

Hence Rolle's theorem is applicable to the function $F(x)$.

(b) For domain of $F(x)$,

$6 - [x] - [x]^2 > 0$ and $x^2 + x - 2 \neq 0$

$\Rightarrow (x + 2)(x - 1) \neq 0 \Rightarrow x \neq -2, 1$

Now $[x]^2 + [x] - 6 < 0 \Rightarrow ([x] + 3)([x] - 2) < 0$

$\Rightarrow -3 < [x] < 2 \Rightarrow -2 \leq x < 2$

$\therefore$ Domain $= (-2, 1) \cup (1, 2)$

(c) We have, $F(\theta) = a \sin\theta + \frac{1}{3}\sin 3\theta$

As $F(\theta)$ has an extremum at $\theta = \frac{\pi}{3}$, so

$a\cos\theta + \cos 3\theta = 0$ at $\theta = \frac{\pi}{3} \Rightarrow \frac{a}{2} - 1 = 0$

$\Rightarrow \frac{a}{2} = 1 \Rightarrow a = 2$

(d) We have $\sum_{k=1}^{2010}\frac{\{x+k\}}{2010} = \frac{\{x+1\}}{2010} + \frac{\{x+2\}}{2010} + \ldots\ldots$

$$+\frac{\{x+2010\}}{2010} = \frac{2010\{x\}}{2010} = \{x\}$$

7. (a, b) : $|z_1 + z_2| = \left|\frac{-b}{a}\right|; z_1 z_2 = \left|\frac{c}{a}\right|$

$\therefore |z_1 + z_2|^2 = 1 \Rightarrow (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = 1$

$\Rightarrow 2 + \bar{z}_1 z_2 + \bar{z}_2 z_1 = 1 \Rightarrow \frac{(z_1 + z_2)^2}{z_1 z_2} = 1$

$\Rightarrow \frac{b^2}{a^2} = \frac{c}{a} \Rightarrow b^2 = ac$

Now, $z_2 = z_1 e^{i\theta}$, then $|z_1 + z_2| = |z_1||1 + e^{i\theta}|$

$\Rightarrow 2\cos\frac{\theta}{2} = 1 \quad \therefore \theta = \frac{2\pi}{3}$

$\therefore PQ = |z_2 - z_1| = \sqrt{3}$

8. (a, b, c) : (a) $\left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$

$= 33 + 11 + 3 + 1 = 48$

(b) $\left[\frac{50}{3}\right] + \left[\frac{50}{3^2}\right] + \left[\frac{50}{3^3}\right] = 16 + 5 + 1 = 22$

(c) $\left[\frac{25}{3}\right] + \left[\frac{25}{3^2}\right] = 8 + 2 = 10$

9. (d) : Let image of A about $y = x$, $y = -2x$ be P and Q.

$\therefore P = (2, 1), Q = \left(\frac{-11}{5}, \frac{2}{5}\right)$

Equation of BC is $x - 7y + 5 = 0$

$\therefore$ In radius $= \perp^r$ distance from $I(0, 0)$ to $BC = \frac{1}{\sqrt{2}}$

10. (a, b) : $I_n = \left(\frac{x^{n+1}}{n+1}\tan^{-1}x\right)_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\frac{1}{1+x^2}dx$

$\Rightarrow (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$

$\Rightarrow (n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx$

$\therefore (n+1)I_n + (n+3)I_{n+2} = \frac{\pi}{2} - \frac{1}{n+2}$

$\therefore a_n = (n + 3) \Rightarrow a_1, a_2, a_3, \ldots$ are in A.P.

$b_n = (n + 1) \Rightarrow b_1, b_2, \ldots$ are in A.P.

$c_n = \frac{\pi}{2} - \frac{1}{n+2} \Rightarrow c_1, c_2, \ldots$ are not in any progression.

11. (a, c) : Orthogonal tangents to a circle meet at the director circle.

$\therefore S_2 \equiv x^2 + y^2 = 2$

Also, $S_3 \equiv x^2 + y^2 = 4$

$\therefore$ Ratio of radius of S_2 and $S_3 = \sqrt{2} : 2 = 1 : \sqrt{2}$

Also, the three circles are concentric.

12. (b, c, d) : Let $(x_1, y_1) = (at^2, 2at)$.

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $(h, (h + at^2)/t)$.

The chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$hx + \left(\frac{h+at^2}{t}\right)y = a^2 \Rightarrow (aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$,

which is a family of straight lines passing through the point of intersection of $ty - a = 0$ and $x + \frac{y}{t} = 0$

So, the fixed point is $(-a/t^2, a/t)$.

Therefore, $x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$

Clearly, $x_1 x_2 = -a^2$, $y_1 y_2 = 2a^2$

Also, $\frac{x_1}{x_2} = -t^4$ and $\frac{y_1}{y_2} = 2t^2$ or $4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$,

13. (b, c) : From the given condition, we have

$2\log_4(2^{1-x} + 1) = \log_2(5\cdot 2^x + 1) + 1$

$\Rightarrow \frac{2\log(2^{1-x} + 1)}{\log 4} = \frac{\log(5\cdot 2^x + 1)}{\log 2} + 1$

$\Rightarrow \log(2^{1-x} + 1) = \log[(5\cdot 2^x + 1)2]$

$\Rightarrow 2^{1-x} + 1 = 10\cdot 2^x + 2$

Put $2^x = y$, so that $\frac{2}{y} + 1 = 10y + 2 \Rightarrow 10y^2 + y - 2 = 0$

$\Rightarrow$ $(5y-2)(2y+1)=0 \Rightarrow y = 2/5$ or $y = -1/2$.
Since $y = 2^x$ cannot be negative,

$\therefore$ $2^x = 2/5 = 0.4$ or $x = \dfrac{\log(2/5)}{\log 2} = 1 - \dfrac{\log 5}{\log 2}$.

14. (a, b, c) : $\int \operatorname{cosec} 2x\,dx = f\{g(x)\} + C$

$\Rightarrow$ $\operatorname{cosec} 2x = f'\{g(x)\}g'(x)$

$\Rightarrow$ $\dfrac{1}{2\tan x} \times \sec^2 x = f'\{g(x)\}g'(x)$

$\Rightarrow$ $f(x) = \dfrac{1}{2x}$, $g'(x) = \sec^2 x$

Domain $f(x) = (-\infty, \infty) - \{0\}$, $g'(x) = \sec^2 x$
Range $g(x) = (-\infty, \infty)$

15. (c) : Here random experiment of arranging 6 persons in a line is $n(S) = 6! = 720$
Let 'a' denotes the person having ₹50 note and 'b' denotes the person having ₹100 note each, since all the six persons, should get ticket. First place should be occupied by a person having ₹50 note and sixth place should be occupied by a person having ₹100 note. So, possible cases are

$\boxed{a}$	$\boxed{aa}$	$\boxed{bb}$	$\boxed{b}$
$\boxed{a}$	$\boxed{ab}$	$\boxed{ab}$	$\boxed{b}$
$\boxed{a}$	$\boxed{ab}$	$\boxed{ba}$	$\boxed{b}$
$\boxed{a}$	$\boxed{ba}$	$\boxed{ab}$	$\boxed{b}$
$\boxed{a}$	$\boxed{ba}$	$\boxed{ba}$	$\boxed{b}$

'a' s can arrange among themselves and 'b' s can arrange among themselves is $n(E) = 3!3! + 3!3! + 3!3! + 3!3! + 3!3! = 180$ ways

$\therefore$ Required probability $= \dfrac{180}{720} = \dfrac{1}{4}$

16. (b) : Let the remaining two observations are x_1 and x_2.

$\therefore$ Mean $(\bar{x}) = \dfrac{2+4+10+12+14+x_1+x_2}{7} = 8$

$\Rightarrow$ $x_1 + x_2 = 14$ ...(i)

And variance $(\sigma^2) = \dfrac{\sum x_i^2}{n} - (\bar{x})^2$

$\Rightarrow$ $16 = \dfrac{4+16+100+144+196+x_1^2+x_2^2}{7} - 64$

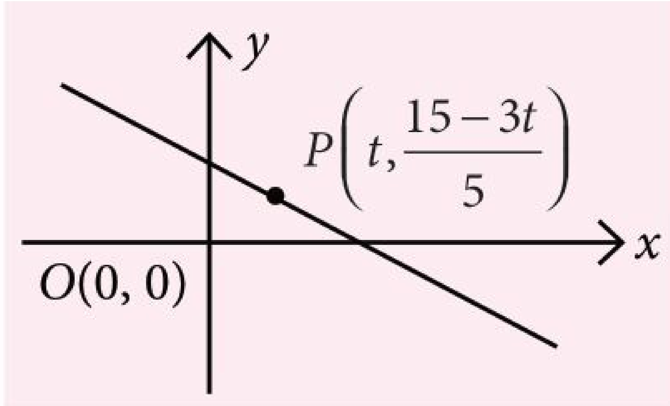
$\Rightarrow$ $80 = \dfrac{460+x_1^2+x_2^2}{7}$

$\Rightarrow$ $x_1^2 + x_2^2 = 560 - 460 = 100$
$\Rightarrow$ $(x_1+x_2)^2 - 2x_1x_2 = 100$

$\Rightarrow$ $x_1x_2 = \dfrac{14^2 - 100}{2}$ [Using (i)]

$\Rightarrow$ $x_1x_2 = 48$

17. (b) : Let $P\left(t, \dfrac{15-3t}{5}\right)$ be any point on the straight line $3x + 5y = 15$.



$\therefore$ $\left|\dfrac{15-3t}{5}\right| = |t|$

$\Rightarrow$ $\dfrac{15-3t}{5} = t$ or $\dfrac{15-3t}{5} = -t$

$\Rightarrow$ $15 - 3t = 5t$ or $15 - 3t = -5t$

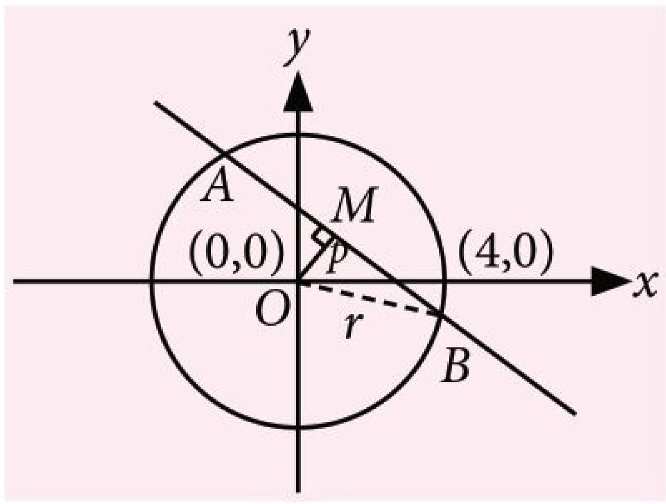
$\Rightarrow$ $15 = 8t$ or $15 = -2t \Rightarrow t = \dfrac{15}{8}$ or $t = \dfrac{-15}{2}$

$\therefore$ Point P lies in 1st or 2nd quadrants.

18. (c) : Let p be the perpendicular distance from $O(0, 0)$ to line $x + y = n$.

$\therefore$ $p = \dfrac{n}{\sqrt{2}} < 4$ for $n = 1, 2, 3, 4, 5$

Now, length of chord



$= 2\sqrt{r^2 - p^2} = 2\sqrt{r^2 - \dfrac{n^2}{2}}$

$= 2\sqrt{16 - \dfrac{n^2}{2}} = \sqrt{64 - 2n^2}$

Now, sum of the squares of the lengths of the chords $= 62 + 56 + 46 + 32 + 14 = 210$

19. (d) : Here, $f(x) = \log_e\left(\dfrac{1-x}{1+x}\right)$

Now, $f\left(\dfrac{2x}{1+x^2}\right) = \log_e\left(\dfrac{1 - \dfrac{2x}{1+x^2}}{1 + \dfrac{2x}{1+x^2}}\right)$

$= \log_e\left(\dfrac{1+x^2-2x}{1+x^2+2x}\right) = \log_e\left(\dfrac{1-x}{1+x}\right)^2$

$= 2\log_e\left(\dfrac{1-x}{1+x}\right) = 2f(x)$

20. (c) : We have, $2y = \left(\cot^{-1}\left(\dfrac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$

$= \left[\cot^{-1}\left(\dfrac{\dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x}{\dfrac{1}{2}\cos x - \dfrac{\sqrt{3}}{2}\sin x}\right)\right]^2 = \left[\cot^{-1}\left(\dfrac{\sin\left(x + \dfrac{\pi}{3}\right)}{\cos\left(x + \dfrac{\pi}{3}\right)}\right)\right]^2$

$$=\left[\cot^{-1}\left(\tan\left(x+\frac{\pi}{3}\right)\right)\right]^2=\left[\frac{\pi}{2}-\tan^{-1}\left(\tan\left(x+\frac{\pi}{3}\right)\right)\right]^2$$

$$=\left[\frac{\pi}{2}-\left(x+\frac{\pi}{3}\right)\right]^2=\left[\frac{\pi}{6}-x\right]^2$$

$$\Rightarrow\ 2\frac{dy}{dx}=2\left(\frac{\pi}{6}-x\right)(-1)\quad\Rightarrow\quad\frac{dy}{dx}=x-\frac{\pi}{6}$$

(21-23) : Let $\vec{r}=x\hat{i}+y\hat{j}$, $x^2+y^2+8x-10y+40=0$, centre $C(-4, 5)$, radius = 1, $p_1=\max\{(x+2)^2+(y-3)^2\}$, $p_2=\min\{(x+2)^2+(y-3)^2\}$

Let $P\equiv(-2, 3)$ and M be any point on circle.

$CP=2\sqrt{2},\ CM=1$

$p_1=(CP+CM)^2=(2\sqrt{2}+1)^2,$

$p_2=(CP-CM)^2=(2\sqrt{2}-1)^2,$

$p_1+p_2=18$

Now, slope of $AB=\left(\frac{dy}{dx}\right)_{(2,\,2)}=-2$

Equation of AB is $2x+y=6$

$\overrightarrow{OA}=2\hat{i}+2\hat{j},\ \overrightarrow{OB}=3\hat{i},\ \overrightarrow{AB}=\hat{i}-2\hat{j},$

$\overrightarrow{AB}\cdot\overrightarrow{OB}=(\hat{i}-2\hat{j})(3\hat{i})=3$

21. (d) 22. (c) 23. (c)

24. (b) : Given, $(1+x+x^2)^n=\sum_{r=0}^{2n}a_r x^r$ ...(i)

Putting $x=1$ in (i), we get

$a_0+a_1+a_2+\ldots+a_{2n}=(1+1+1)^n=3^n$

But $a_r=a_{2n-r}$ for $0\le r<n-1$.

$\therefore\ 2(a_0+a_1+\ldots+a_{n-1})+a_n=3^n$

25. (a) : Replacing x by $-1/x$ in (i), we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n=\sum_{r=0}^{2n}(-1)^r\frac{a_r}{x^r}$$

Now, $a_0^2-a_1^2+a_2^2-a_3^2+\ldots+a_{2n}^2$ = coefficient of the constant term in $(a_0+a_1x+\ldots+a_{2n}x^{2n})$

$$\left[a_0-\frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+\ldots+\frac{a_{2n}}{x^{2n}}\right]$$

= coefficient of the constant term in

$$(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$$

= coefficient of the constant term in

$$\frac{(1+x+x^2)^n(x^2-x+1)^n}{x^{2n}}$$

= coefficient of x^{2n} in $[(x^2+1)^2-x^2]^n$

= coefficient of x^{2n} in $(1+x^2+x^4)^n$

= coefficient of y^n in $(1+y+y^2)^n=a_n$

26. (b) : $\sum_{r=0}^{n}(-1)^r a_r\,{}^nC_r$ = coefficient of the constant term in $[a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}]$

$$\times\left[{}^nC_0-{}^nC_1\left(\frac{1}{x}\right)+{}^nC_2\left(\frac{1}{x}\right)^2+\ldots+(-1)^n\,{}^nC_n\left(\frac{1}{x}\right)^n\right]$$

= coefficient of the constant term in $(1+x+x^2)^n\left(1-\frac{1}{x}\right)^n$

= coefficient of x^n in $(x^2+x+1)^n(x-1)^n$

= coefficient of x^n in $(x^3-1)^n$

= 0 as n is not a multiple of 3.

(27-28) : $\sin(x+y)>0$

$\Rightarrow x+y\in(0,\pi)\cup(2\pi, 3\pi)$ etc.

and $\sin(x+y)<0$

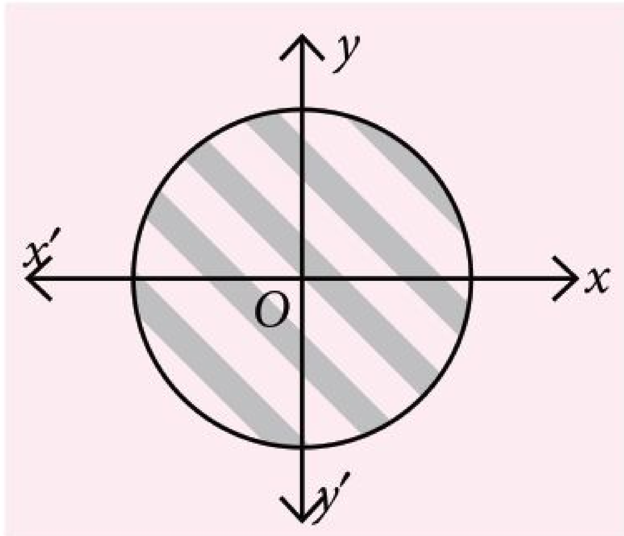
$\Rightarrow x+y\in(\pi, 2\pi)\cup(3\pi, 4\pi)$ etc.

Hence, $A_1=A_2$

and A_1+A_2= Full circle area

$=100\pi$

Hence, $A_1=A_2=50\pi$



27. (c) 28. (b)

29. (c) : Here $x=\frac{1}{1-\sin^2\theta}\ \Rightarrow\ \cos^2\theta=\frac{1}{x}$

and $y=\frac{1}{1-\cos^2\theta}\ \Rightarrow\ \sin^2\theta=\frac{1}{y}$

(A) $\sum_{n=0}^{\infty}\sin^{2n}\theta\cos^{2n}\theta=\frac{1}{1-\sin^2\theta\cos^2\theta}$

$$=\frac{1}{1-\frac{1}{xy}}=\frac{xy}{xy-1}$$

(B) $\sum_{n=0}^{\infty}\tan^{2n}\theta=\frac{1}{1-\tan^2\theta}=\frac{1}{1-\frac{x}{y}}=\frac{y}{y-x}$

(C) $\sum_{n=0}^{\infty}\sin^{2n}\theta\cos^{4n}\theta=\frac{1}{1-\sin^2\theta\cos^4\theta}$

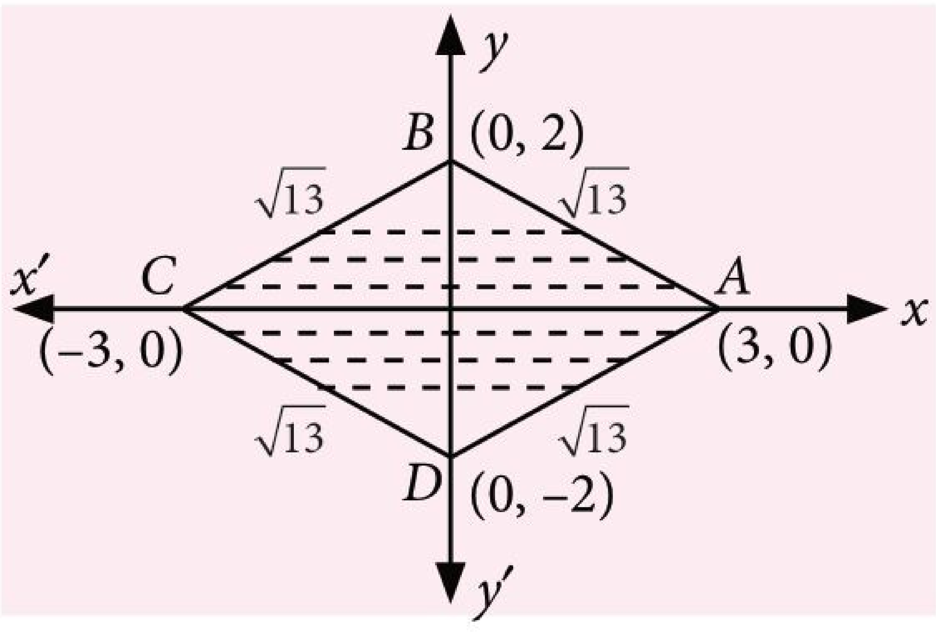
$$=\frac{1}{1-\frac{1}{x^2y}}=\frac{x^2y}{x^2y-1}$$

(D) $\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{4n}\theta=\frac{1}{1-\cos^2\theta\sin^4\theta}=\frac{1}{1-\frac{1}{xy^2}}$

$$=\frac{xy^2}{xy^2-1}.$$

30. (a) : (A) $\because d(x, y) = 2|x| + 3|y| = 6$ (given)

$\Rightarrow \frac{|x|}{3}+\frac{|y|}{2}=1$



$\therefore$ Perimeter,

$\lambda=4\sqrt{13}$

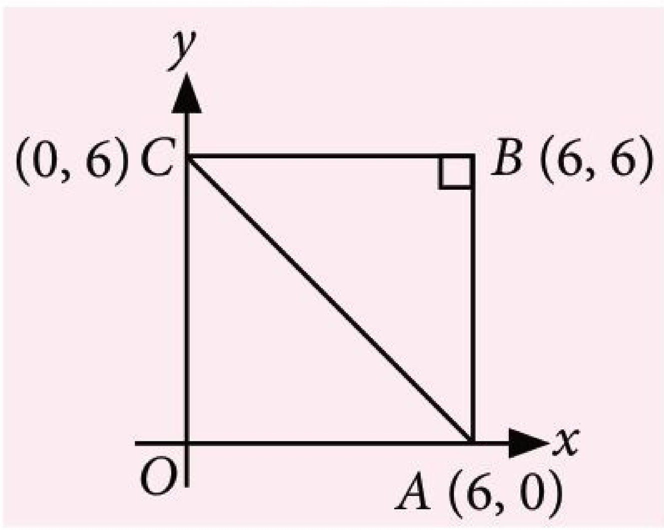
and area,

$\mu=4\times\frac{1}{2}\times3\times2=12$

Thus, $\frac{\lambda^2}{16}-\mu=1$ and $\lambda^2 - \mu^2 = 64$

(B) It is clear that orthocentre is (6, 6)

Circumcentre $\equiv (3, 3)$ and centroid $\equiv (4, 4)$

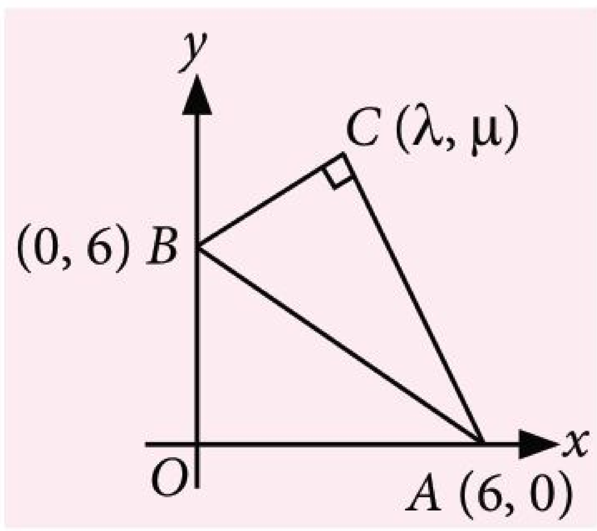


$\therefore\ \lambda=\sqrt{(6-3)^2+(6-3)^2}$
$=\sqrt{9+9}=3\sqrt{2}$

and $\mu=\sqrt{(4-3)^2+(4-3)^2}$
$=\sqrt{1+1}=\sqrt{2}$

$\therefore\ \lambda^2 - \mu^2 = 16$

(C) Slope of AC × Slope of $BC = -1$



$\Rightarrow \left(\frac{\mu-0}{\lambda-6}\right)\times\left(\frac{\mu-6}{\lambda-0}\right)=-1$

$\Rightarrow \mu^2 - 6\mu = -\lambda^2 + 6\lambda$

$\Rightarrow \lambda^2 + \mu^2 - 6\lambda - 6\mu = 0$

31. (d) : (A) $C_1 : x^2 + y^2 - 2a(x + y) + a^2 = 0$,

Centre : (a, a), radius : a,

$C_2 : x^2 + y^2 - 2b(x + y) + b^2 = 0$

Centre : (b, b), radius b

Since C_1 and C_2 touch each other

$\Rightarrow \sqrt{2}(b-a) = a+b \quad \Rightarrow \frac{b}{a} = (\sqrt{2}+1)^2 = 3+2\sqrt{2}$

(B) C_1 and C_2 intersect orthogonally

$\Rightarrow 2(b - a)^2 = a^2 + b^2 \quad \Rightarrow \frac{b}{a}=2+\sqrt{3}$

(C) The common chord is

$2(b - a)(x + y) = b^2 - a^2$.

It passes through $(a, a) \Rightarrow b/a = 3$.

(D) C_2 passes through (a, a)

$\Rightarrow 2a^2-4ab+b^2=0 \Rightarrow \frac{b}{a}=2+\sqrt{2}$.

32. (2) : Since two parabolas are symmetrical about $y = x$.

Solving $y=x$ and $y^2-4x-8y+40=0$, we get $x^2 - 12x + 40 = 0$ has no real solution.

$\therefore$ They don't intersect.

Point on $(x-4)^2=4(y-6)$ is (6, 7) and the corresponding point on $(y-4)^2=4(x-6)$ is (7, 6). Minimum distance is $\sqrt{2}$.

33. (0.5) : We have, $\lim\limits_{x\to\infty}\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}$

$$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{\sqrt{x^3}}}}+1}=\frac{1}{2}$$

$= 0.5$

34. (2) : L.C.M. $(p, q) = 2^2\cdot3^4\cdot5^2$

Let $p=2^{a_1}\cdot3^{b_1}\cdot5^{c_1}$ and $q=2^{a_2}\cdot3^{b_2}\cdot5^{c_2}$

$\Rightarrow \max\{a_1, a_2\} = 2 \Rightarrow$ 5 ways

$\Rightarrow \max\{b_1, b_2\} = 4 \Rightarrow$ 9 ways

$\Rightarrow \max\{c_1, c_2\} = 2 \Rightarrow$ 5 ways

$\therefore\ K = 3^2 \cdot 5^2$ can be expressed as $1\cdot3^2\cdot5^2$, $3^2\cdot5^2$

35. (15) : $\frac{r}{R}=\cos A+\cos B+\cos C-1$

$\Rightarrow \frac{1}{8}=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-1+\cos C$

$\Rightarrow \frac{1}{8}=\sin\frac{C}{2}-2\sin^2\frac{C}{2} \Rightarrow \sin\frac{C}{2}=\frac{1}{4}$

$\therefore\ \cos C=1-\frac{1}{8}=\frac{7}{8}$. So, $\frac{1+\cos C}{1-\cos C}=\frac{1+7/8}{1-7/8}=15$

36. (5) : Image of orthocentre of ΔABC w.r.t. BC lies on the circle.

37. (1155) : Given, period of $f(x)$ is 7

$\therefore$ Period of $f\left(\frac{x}{3}\right)$ is $\frac{7}{1/3}=21$

and period of $g(x)$ is 11

$\therefore$ Period of $g\left(\frac{x}{5}\right)$ is $\frac{11}{1/5}=55$

Now, T_1 = Period of $f(x)\,g\left(\frac{x}{5}\right)=7\times55=385$

and T_2 = Period of $g(x)f\left(\frac{x}{3}\right)=11\times21=231$

$\therefore$ Period of $F(x)$ = L.C.M. of $\{T_1, T_2\}$

= L.C.M. of {385, 231}

= 7 × 11 × 3 × 5 = 1155

38. (60) : Required number of ways $=\dfrac{4!}{2!2!}\times\dfrac{5!}{2!3!}$ $= 60$

39. (18) : The equation of the line L be $y - 2 = m\,(x - 8)$, $m < 0$. Coordinates of P and Q are $P\left(8-\dfrac{2}{m},0\right)$ and $Q\,(0, 2 - 8m)$.

Now, $OP + OQ = 8-\dfrac{2}{m}+2-8m=10+\dfrac{2}{(-m)}+8(-m)$

$\geq 10+2\sqrt{\dfrac{2}{(-m)}\times 8(-m)}\geq 18$

So, absolute minimum value of $OP + OQ = 18$

40. (4) : We have $f(x) = \begin{cases} x-1, & 1\leq x<2 \\ 1-x, & 0\leq x<1 \end{cases}$

$f(x)$ is periodic with period 2 and also it is an even function.

Also, $\cos \pi x$ has period 2

$\therefore\ I = \displaystyle\int_{-10}^{10} f(x)\cos \pi x\,dx = 2\int_{0}^{10} f(x)\cos \pi x\,dx$

($\because$ $f(x)\cos \pi x$ is an even function)

$= 2\times 5\displaystyle\int_0^2 f(x)\cos \pi x\,dx$

$= 10\left[\displaystyle\int_0^1(1-x)\cos \pi x\,dx+\int_1^2(x-1)\cos \pi x\,dx\right]$

$= 10(I_1 + I_2)$

$I_2 = \displaystyle\int_1^2 (x-1)\cos \pi x\,dx$

$I_2 = -\displaystyle\int_0^1 t\cos \pi t\,dt$ (put $x - 1 = t$)

$I_1 = \displaystyle\int_0^1 (1-x)\cos \pi x\,dx = -\int_0^1 x\cos(\pi x)\,dx$

$\therefore\ I = 10\left[-2\displaystyle\int_0^1 x\cos \pi x\,dx\right] = -20\left[x\dfrac{\sin \pi x}{\pi}+\dfrac{\cos \pi x}{\pi^2}\right]_0^1$

$= -20\left[-\dfrac{1}{\pi^2}-\dfrac{1}{\pi^2}\right] = \dfrac{40}{\pi^2}$

$\therefore\ \dfrac{\pi^2}{10}I = 4$

◈◈

CBSE warm-up!

CLASS-XI

TERM-I OBJECTIVE TYPE QUESTIONS*

Series 3 — Complex Numbers and Quadratic Equations

MCQs

1. Express $(4-3i)^3$ in the standard form.
   (a) $-44-177i$ (b) $44-177i$
   (c) $44+177i$ (d) $-44+177i$

2. $i^{57}+\dfrac{1}{i^{125}}$ is equal to
   (a) $2i$ (b) $-2i$ (c) 0 (d) 2

3. Find the multiplicative inverse of $2-3i$.
   (a) $2/13+i(3/13)$ (b) $2/13-i(3/13)$
   (c) $2+3i$ (d) None of these

4. Solve : $x^2+x+1=0$
   (a) $\pm i$ (b) $\pm(1/2)i$
   (c) $\dfrac{-1\pm\sqrt{3}i}{2}$ (d) $\dfrac{\pm\sqrt{3}i}{2}$

5. What is the reciprocal of $3+\sqrt{7}i$?
   (a) $\dfrac{3}{16}-\dfrac{1}{16}i$ (b) $\dfrac{3}{16}+\dfrac{1}{16}i$
   (c) $\dfrac{3}{16}+\dfrac{\sqrt{7}}{16}i$ (d) $\dfrac{3}{16}-\dfrac{\sqrt{7}}{16}i$

6. The argument of the complex number $\left(\dfrac{i}{2}-\dfrac{2}{i}\right)$ is equal to
   (a) $\dfrac{\pi}{4}$ (b) $\dfrac{3\pi}{4}$ (c) $\dfrac{\pi}{12}$ (d) $\dfrac{\pi}{2}$

7. If $(x+iy)^{1/3}=a+ib$, where $x, y, a, b \in R$, then $\dfrac{x}{a}-\dfrac{y}{b}=$
   (a) a^2-b^2 (b) $-2(a^2+b^2)$
   (c) $2(a^2-b^2)$ (d) a^2+b^2

8. $i^2+i^4+i^6+\ldots.\ (2n+1)$ terms =
   (a) 1 (b) -1 (c) i (d) $-i$

9. Find the modulus of $\dfrac{1+i}{1-i}$.
   (a) 1 (b) 2 (c) -1 (d) -2

10. If $1-i$ is a root of the equation $x^2+ax+b=0$, where $a, b\in R$, then find the value of a.
    (a) 2 (b) -2 (c) 3 (d) -3

11. Simplify : $i^{n+100}+i^{n+50}+i^{n+48}+i^{n+46}$
    (a) 0 (b) 1 (c) 2 (d) 3

12. Express in standard form: $(1+2i)^6(4i+3)^3$
    (a) -14225 (b) -15625
    (c) -18625 (d) -17525

13. Find the sum of the complex number $-\sqrt{3}+\sqrt{-2}$ and $2\sqrt{3}-i$.
    (a) $\sqrt{3}+i(\sqrt{2}-1)$ (b) $\sqrt{3}-i(\sqrt{2}-1)$
    (c) $\sqrt{3}-i(\sqrt{2}+1)$ (d) None of these

14. Write the additive inverse of $-5+4i$.
    (a) $5+4i$ (b) $5-4i$
    (c) $5+3i$ (d) $5-3i$

15. Find additive inverse of $\dfrac{3}{\sqrt{2}+i}$.
    (a) $\sqrt{2}-i$ (b) $-\sqrt{2}+i$
    (c) $\sqrt{2}+i$ (d) $-\sqrt{2}-i$

*Chapterwise practice questions for CBSE Exam Term- I as per the pattern issued by CBSE.

16. Evaluate: $i^{141} + i^{142} + i^{143} + i^{144}$
(a) 0 (b) 1 (c) 2 (d) 3

17. If $\left(\frac{1+i}{1-i}\right)^n = 1$, then find the least positive integral value of n.
(a) 0 (b) 1 (c) 3 (d) 4

18. Evaluate : $(-\sqrt{-1})^{4n+3}, n \in N$
(a) 0 (b) 1 (c) i (d) $-i$

19. Evaluate : $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2$
(a) 2 (b) -2 (c) 4 (d) -4

20. What is the smallest positive integer n, for which $(1 + i)^{2n} = (1 - i)^{2n}$?
(a) 2 (b) 4 (c) 6 (d) 8

21. Find the multiplicative inverse of $\sqrt{5} + 3i$.
(a) $\frac{\sqrt{5}}{14} + \frac{3}{\sqrt{14}}i$ (b) $-\frac{\sqrt{5}}{14} - \frac{3}{14}i$
(c) $\frac{\sqrt{5}}{14} - \frac{3}{14}i$ (d) None of these

22. Multiply $3 - 2i$ by its conjugate.
(a) 13 (b) 17 (c) 21 (d) 23

23. Find the conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.
(a) $\frac{63}{25} + \frac{16}{25}i$ (b) $\frac{63}{25} - \frac{16}{25}i$
(c) $-\frac{63}{25} - \frac{16}{25}i$ (d) None of these

24. If $z = 2 + \sqrt{3}i$, find the value of $z \cdot \bar{z}$.
(a) 5 (b) 7 (c) 9 (d) 11

25. For every complex number $z = x + iy \neq 0 + i \cdot 0$; there exists a complex number $z_1 = \frac{x - iy}{x^2 + y^2}$, then value of zz_1 equals
(a) 0 (b) 1 (c) 2 (d) 3

26. Write the complex number $z = \frac{2+i}{(1+i)(1-2i)}$ in standard form.
(a) $\frac{1}{2} + \frac{i}{2}$ (b) $\frac{1}{2} - \frac{i}{2}$
(c) $-\frac{1}{2} + \frac{i}{2}$ (d) $-\frac{1}{2} - \frac{i}{2}$

27. Roots of the equation $x^2 - 4x + 13 = 0$ by factorisation methods are
(a) $7 \pm 11i$ (b) $2 \pm 3i$
(c) $3 \pm 2i$ (d) $5 \pm 7i$

28. Express $(5 - 3i)^3$ in the form of $a + ib$.
(a) $0 + 198i$ (b) $-10 - 198i$
(c) $10 - 198i$ (d) $10 + 198i$

29. The arg $\left(\frac{3+i}{2-i} + \frac{3-i}{2+i}\right)$ is equal to
(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $-\frac{\pi}{4}$

30. The real values of x and y for which the following equality hold, are respectively
$(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy)$
(a) 2, 3 or -2, 1/3 (b) 1, 3 or -1, 1/3
(c) 2, 1/3 or -2, 3 (d) 2, 1/3 or -2, $-1/3$

31. $(-i)(2i)\left(-\frac{1}{8}i\right)^3$ equals
(a) $0 + i$ (b) $0 - i$
(c) $\frac{1}{256} + 0i$ (d) $0 + \frac{1}{256}i$

32. If $|z^2 - 1| = |z|^2 + 1$, then z lies on
(a) imaginary axis (b) real axis
(c) origin (d) None of these

33. Solve : $x^2 + 3 = 0$
(a) $\pm\sqrt{3}\,i$ (b) $\pm\sqrt{2}\,i$
(c) $\pm\sqrt{5}\,i$ (d) $\pm\sqrt{7}\,i$

34. Solve : $x^2 - 14x + 58 = 0$
(a) $7 \pm 3i$ (b) $5 \pm 2i$
(c) $7 \pm 2i$ (d) $5 \pm 3i$

35. Solve : $5x^2 - 6x + 2 = 0$
(a) $\frac{3 \pm 2i}{5}$ (b) $\frac{3 \pm i}{5}$
(c) $\frac{3 \pm 5i}{2}$ (d) $\frac{3 \pm i}{9}$

36. Write the roots of the quadratic equation $x^2 + 8 = 0$.
(a) $\pm 2\sqrt{2}i$ (b) $\pm 3\sqrt{3}i$
(c) $\pm 3\sqrt{2}\,i$ (d) $\pm 2\sqrt{3}\,i$

37. Solve for $x : x^2 + 3x + 9 = 0$.
(a) $\frac{-3 \pm 2\sqrt{3}\,i}{2}$ (b) $\frac{-3 \pm 2\sqrt{2}\,i}{5}$
(c) $\frac{-3 \pm 3\sqrt{3}\,i}{2}$ (d) None of these

38. If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y.
(a) 1/4, 3/4 (b) 3/4, 33/4
(c) 33/4, −3/4 (d) −3/4, −33/4

39. If $x = 2 + 5i$, then value of the expression $x^3 - 5x^2 + 33x - 49$ equals
(a) −20 (b) 10
(c) 20 (d) −29

40. If $(x + iy)(p + iq) = (x^2 + y^2)i$, then
(a) $p = x^2, q = y^2$ (b) $p = y, q = x$
(c) $p = x, q = y$ (d) $p = -x, q = y$

41. $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$ equals
(a) $1-2\sqrt{2}i$ (b) $1-\sqrt{2}i$
(c) $1+2\sqrt{2}i$ (d) $1+\sqrt{2}i$

42. Convert $\dfrac{1+7i}{(2-i)^2}$ into standard form.
(a) $0 + i$ (b) $1 - i$
(c) $-2 - 3i$ (d) $-1 + i$

43. If $z = 2 + i$, then $(z-1)(\bar{z}-5)+(\bar{z}-1)(z-5)$ is equal to
(a) 2 (b) 7 (c) − 1 (d) − 4

44. $\left|(1+i)\left(\dfrac{2+i}{3+i}\right)\right|$ is equal to
(a) $-\dfrac{1}{2}$ (b) 1 (c) − 1 (d) $\dfrac{1}{2}$

45. If α and β are different complex numbers with $|\beta| = 1$, then $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ is
(a) 0 (b) 3/2 (c) 1/2 (d) 1

46. If $2 + i$ is a root of equation $x^3 - 5x^2 + 9x - 5 = 0$, then the other roots are
(a) 1 and $2 - i$ (b) − 1 and $3 + i$
(c) 0 and 1 (d) − 1 and $i - 2$

47. Find the value of P such that the difference of the roots of the equation $x^2 - Px + 8 = 0$ is 2.
(a) ±6 (b) ±3 (c) ±5 (d) ±4

48. If $1 - i$ is a root of the equation $x^2 + ax + b = 0$, where $a, b \in R$, then find the values of a and b.
(a) 2, 2 (b) −2, 2 (c) −2, −2 (d) 1, 2

49. Number of solutions of the equation $z^2 + |z|^2 = 0$ is
(a) 1 (b) 2
(c) 3 (d) infinitely many

50. The equation of smallest degree with real coefficients having $2 + 3i$ as one of the roots is
(a) $x^2 + 4x + 13 = 0$ (b) $x^2 + 4x - 13 = 0$
(c) $x^2 - 4x + 13 = 0$ (d) $x^2 - 4x - 13 = 0$

CASE BASED

Case-I : Read the following and answer any four questions from 51 to 55 given below.

On Sunday, Kanika was playing a ludo game with her mother. Kanika threw a die on the ludo cardboard. If the die stopped at a point (−3, 4), then answer the following questions.

51. Let the point is represented as z in an argand plane, then find its modulus.
(a) $\sqrt{7}$ (b) 6 (c) $\sqrt{17}$ (d) 5

52. The multiplicative inverse of z is
(a) $\dfrac{-3}{25}+4i$ (b) $\dfrac{-3}{25}+\dfrac{4}{25}i$
(c) $\dfrac{-3}{25}-\dfrac{4}{25}i$ (d) $-3+\dfrac{4}{25}i$

53. If her mother throw a die at $(1, \sqrt{3})$ which is represented by z' in a argand plane, then $z' \cdot z =$
(a) $1+i(4-3\sqrt{3})$
(b) $(-3+4\sqrt{3})+i(4-3\sqrt{3}i)$
(c) $(-3-4\sqrt{3})+i(4-3\sqrt{3})$
(d) $(-3+4\sqrt{3})+i$

54. Find modulus of z'.
(a) 2 (b) 1 (c) 3 (d) 4

55. The value of conjugate of $z + z'$ is
(a) $-2+(4+\sqrt{3})i$ (b) $-2 + i$
(c) $-2 + 4i$ (d) $-2-(4+\sqrt{3})i$

Case-II : Read the following and answer any four questions from 56 to 60 given below.

Rohan and Rohit both are studying in Class XI. During summer vacations their father who is a mathematics teacher gave them a problem of quadratic equation to solve. Rohan found that the sum and product of roots were 3 and 9 respectively, whereas Rohit found that the sum and product of roots were −3 and −9 respectively.

Their father told them, Rohan found a correct product and Rohit found a correct sum.

56. The correct quadratic equation is
(a) $x^2 - 3x + 9 = 0$ (b) $x^2 + 3x - 9 = 0$
(c) $x^2 + 3x + 9 = 0$ (d) $x^2 - 9x + 3 = 0$

57. The discriminant of quadratic equation is
(a) equal to zero (b) less than zero
(c) greater than zero (d) can't compare

58. The roots of quadratic equation are
(a) $\frac{-3\pm 3\sqrt{3}i}{2}$ (b) $\frac{1\pm\sqrt{3}i}{2}$
(c) $\frac{-3\pm\sqrt{3}i}{2}$ (d) $\frac{3\pm\sqrt{3}i}{2}$

59. If $z_1 = \frac{-3+3\sqrt{3}i}{2}$ and $z_2 = \frac{-3-3\sqrt{3}i}{2}$, then $|z_1z_2| =$
(a) 9 (b) 5 (c) 4 (d) 6

60. Complex roots of quadratic equation are always
(a) negative of each other
(b) same
(c) occur in conjugate pair
(d) can't say

ASSERTION & REASON

Directions (Q.No. 61-70) : In the following questions, a statement of assertion (Statement-I) is followed by a statement of reason (Statement-II). Mark the correct choice as :
(a) If both Statement-I and Statement-II are true and Statement-II is the correct explanation of Statement-I.
(b) If both Statement-I and Statement-II are true but Statement-II is not the correct explanation of Statement-I.
(c) If Statement-I is true but Statement-II is false.
(d) If Statement-I is false and Statement-II is true.

61. **Statement-I :** Consider $|z_1| = 1$, $|z_2| = 2$ and $|z_3| = 3$. If $|z_1 + 2z_2 + 3z_3| = 6$, then the value of $|z_2z_3 + 8z_3z_1 + 27z_1z_2|$ is 36.
Statement-II : $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.

62. **Statement-I :** Let $f(x)$ be a quadratic expression such that $f(0) + f(1) = 0$. If -2 is one of the roots of $f(x) = 0$, then other root is 3/5.
Statement-II : If α and β are the zeroes of $f(x) = ax^2 + bx + c$, then sum of zeroes $= -b/a$ and product of zeroes $= c/a$.

63. **Statement-I :** If P and Q are the points in the argand plane representing the complex numbers z_1 and z_2 respectively, then distance $|PQ| = |z_2 - z_1|$.
Statement-II : Locus of the point $P(z)$ satisfying $|z - (2 + 3i)| = 4$ is a straight line.

64. **Statement-I :** If $z \ne 0$ is a complex number such that $\arg z = \pi/4$, then $\text{Re}(z^2) = 0$.
Statement-II : If $z \ne 0$ and $\arg z = \pi/4$, then $\text{Re}\, z = -\text{Im}\, z$.

65. **Statement-I :** The equation $3x^2 - 3x + 2 = 0$ has non-real roots.
Statement-II : If a, b, c are real and $b^2 - 4ac \ge 0$, then the roots of the equation $ax^2 + bx + c = 0$ are real and if $b^2 - 4ac < 0$, then roots of $ax^2 + bx + c = 0$ are non-real.

66. **Statement-I :** Expression $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$ can be expressed in the form of $\frac{22}{17}+\frac{3}{17}i$.
Statement-II : Any number of the form $z = a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called complex number.

67. **Statement-I :** The conjugate of $\frac{(1+i)^2}{3-i}$ is $-\frac{1}{5}+\frac{3}{5}i$.
Statement-II : Conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and defined as $\bar{z} = a - ib$.

68. **Statement-I :** The argument of complex number $-2+2\sqrt{3}i$ is $\frac{2\pi}{3}$.
Statement-II : Modulus of a complex number $z = a + ib$ is denoted by $|z|$ and defined as $|z| = \sqrt{a^2+b^2}$.

69. **Statement-I :** If amplitude of $(z - 2 - 3i)$ is $\frac{\pi}{4}$, then locus of z represents a circle.
Statement-II : For any three complex numbers z_1, z_2 and z_3, $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

70. **Statement-I :** The real and imaginary part of the complex number $\sqrt{37}+\sqrt{-19}$ are $\sqrt{37}$ and $-\sqrt{19}$ respectively.
Statement-II : Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$.

SOLUTIONS

1. **(a) :** We have, $(4 - 3i)^3$
$= 4^3 - (3i)^3 - 3 \times 4 \times 3i \times (4 - 3i)$
$= 64 - 27i^3 - 36i(4 - 3i) = (64 - 108) + i(27 - 144)$
$= (-44 - 117i)$
$\therefore \quad (4 - 3i)^3 = (-44 - 117i)$

2. **(c) :** Since $i^{4n} = 1$ and $i^{4n+1} = i$ $(n \in N)$

$\therefore \quad i^{57} + \frac{1}{i^{125}} = i + \frac{1}{i} = i - i = 0$

3. **(a) :** Let $z = 2 - 3i$, then $z^{-1} = \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$

$= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$

4. **(c) :** Here, $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = 1 - 4 = -3$

Therefore, the solutions are given by

$x = \frac{-1 \pm \sqrt{-3}}{2 \times 1} = \frac{-1 \pm \sqrt{3}i}{2}$

5. **(d) :** Reciprocal of $z = \frac{\overline{z}}{|z|^2}$

Therefore, reciprocal of $3 + \sqrt{7}i = \frac{3-\sqrt{7}i}{16} = \frac{3}{16} - \frac{\sqrt{7}}{16}i$

6. **(d) :** Since $\left(\frac{i}{2} - \frac{2}{i}\right) = \frac{i}{2} - \frac{2i}{i^2} = \frac{i}{2} + 2i = \frac{5}{2}i$

So, argument of $\left(\frac{5}{2}i\right) = \tan^{-1}\left(\frac{5/2}{0}\right) = \frac{\pi}{2}$.

7. **(b) :** $(x + iy)^{1/3} = a + ib \Rightarrow x + iy = (a + ib)^3$

$\Rightarrow \quad x + iy = a^3 - ib^3 + i3a^2b - 3ab^2$

$= a^3 - 3ab^2 + i(3a^2b - b^3)$

$\therefore \quad x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

So, $\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2$

$= -2a^2 - 2b^2 = -2(a^2 + b^2)$

8. **(b) :** The first $2n$ terms will cancel in pairs and the last term is $(i^2)^{2n+1} = i^{4n} \cdot i^2 = -1$

9. **(a) :** Let $z = \frac{1+i}{1-i}$.

Then, $|z| = \left|\frac{1+i}{1-i}\right| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{1^2+1^2}}{\sqrt{1^2+1^2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$

10. **(b) :** Given $1 - i$ is a root of the equation $x^2 + ax + b = 0$, where $a, b \in R$, therefore $1 + i$ will also be the root of the given equation.

Now, sum of roots $= (1 - i) + (1 + i) = -a/1 \Rightarrow a = -2$

11. **(a) :** Given expression $= i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46}$

$= i^n(i^{100} + i^{50} + i^{48} + i^{46})$

$= i^n[(i^2)^{50} + (i^2)^{25} + (i^2)^{24} + (i^2)^{23}]$

$= i^n[(-1)^{50} + (-1)^{25} + (-1)^{24} + (-1)^{23}]$

$= i^n[1 - 1 + 1 - 1] = i^n \cdot 0 = 0$

12. **(b) :** We have, $(1 + 2i)^6(4i + 3)^3$

$= ((1 + 2i)^3)^2(4i + 3)^3 = (1 + 8i^3 + 6i + 12i^2)^2$

$(64i^3 + 27 + 108i + 144i^2)$

$= (1 - 12 + 6i - 8i)^2 (-64i + 108i + 27 - 144)$

$= (-11 - 2i)^2(44i - 117) = 1936i^2 - 13689 = -15625$

13. **(a) :** Let $z_1 = -\sqrt{3} + \sqrt{-2} = -\sqrt{3} + i\sqrt{2}$

and $z_2 = 2\sqrt{3} - i$

Then $z_1 + z_2 = (-\sqrt{3} + i\sqrt{2}) + (2\sqrt{3} - i)$

$= (-\sqrt{3} + 2\sqrt{3}) + i(\sqrt{2} - 1) = \sqrt{3} + i(\sqrt{2} - 1)$

14. **(b) :** Let $z = -5 + 4i$

The additive inverse of z is $-z$ *i.e.* $-(-5 + 4i) = 5 - 4i$.

15. **(b) :** Let $z = \frac{3}{\sqrt{2}+i} = \frac{3(\sqrt{2}-i)}{2+1} = \sqrt{2} - i$

$\therefore$ Additive inverse of z is $-z$ *i.e.* $-\sqrt{2} + i$.

16. **(a) :** We have, $i^{141} + i^{142} + i^{143} + i^{144}$

$= i^{140}[i + i^2 + i^3 + i^4]$

$= (i^4)^{35}[i - 1 - i + 1]$ $\quad [\because i^2 = -1, i^3 = -i, i^4 = 1]$

$= 0$.

17. **(d) :** We have, $\left(\frac{1+i}{1-i}\right)^n = 1$

$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = 1$

$\Rightarrow \left(\frac{1-1+2i}{1+1}\right)^n = 1 \Rightarrow \left(\frac{2i}{2}\right)^n = 1 \Rightarrow i^n = 1$

$\Rightarrow$ n is multiple of 4.

$\therefore$ The least positive integral value of n is 4.

18. **(c) :** $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3}$ $\quad [\because \sqrt{-1} = i]$

$= (-i)^{4n}(-i)^3 = 1(-1)(-i)$ $\quad [\because i^4 = 1, i^3 = -i]$

$= i$

19. **(d) :** Given expression

$\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{18} \cdot i + \left(\frac{1}{i}\right)^{25}\right]^2$

$= [(i^2)^9 \cdot i + (-i)^{25}]^2 = [(-1)^9 \cdot i - i^{25}]^2 = [-i - i^{24} \cdot i]^2$

$= [-i - (i^2)^{12} \cdot i]^2 = [-i - (-1)^{12} i]^2$

$= (-i - i)^2 = (-2i)^2 = 4i^2 = -4$

20. **(a) :** We have, $(1 + i)^{2n} = (1 - i)^{2n}$

$\Rightarrow \left(\frac{1+i}{1-i}\right)^{2n} = 1 \Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^{2n} = 1$

$\Rightarrow \left(\frac{1-1+2i}{1+1}\right)^{2n} = 1 \Rightarrow \left(\frac{2i}{2}\right)^{2n} = 1$

$\Rightarrow$ $(i)^{2n} = 1 \Rightarrow 2n$ is a multiple of 4.

$\therefore$ The least positive integral value of n is 2.

21. (c) : Let $z=\sqrt{5}+3i$

$\bar{z}=\sqrt{5}-3i$ and $|z|^2=(\sqrt{5})^2+(3)^2=14$

$$z^{-1}=\frac{\bar{z}}{|z|^2}=\frac{\sqrt{5}-3i}{14}=\frac{\sqrt{5}}{14}-\frac{3}{14}i$$

22. (a) : Let $z = 3 - 2i$

$\bar{z}=3+2i$

$\therefore \quad z\cdot\bar{z}=(3-2i)(3+2i)=9-4i^2=9+4=13$

23. (a) : Let $z=\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}=\frac{6+9i-4i+6}{2-i+4i+2}$

$$=\frac{12+5i}{4+3i}=\left(\frac{12+5i}{4+3i}\right)\times\left(\frac{4-3i}{4-3i}\right)$$

$$=\frac{48-36i+20i+15}{4^2-(3i)^2}=\frac{63-16i}{16+9}=\frac{63}{25}-\frac{16}{25}i$$

$\therefore$ Conjugate of $z=\frac{63}{25}+\frac{16}{25}i.$

24. (b) : $z=2+\sqrt{3}i$

$\therefore \quad \bar{z}=2-\sqrt{3}i$

$z\cdot\bar{z}=(2+\sqrt{3}i)(2-\sqrt{3}i)=4-3i^2=4+3=7$

25. (b) : $zz_1=(x+iy)\left(\frac{x-iy}{x^2+y^2}\right)$

$$=(x+iy)\left(\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}\right)$$

$$=\left(\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}\right)+i\left(\frac{-xy}{x^2+y^2}+\frac{xy}{x^2+y^2}\right)$$

$= 1 + i\cdot(0) = 1$

26. (a) : We have,

$$z=\frac{2+i}{(1+i)(1-2i)}=\frac{2+i}{3-i}=\frac{(2+i)(3+i)}{(3-i)(3+i)}$$

$$=\frac{5+5i}{10}=\frac{1}{2}+i\frac{1}{2}$$

27. (b) : We have, $x^2 - 4x + 13 = 0$

$$\therefore \quad x=\frac{4\pm\sqrt{16-4(13)}}{2}=\frac{4\pm\sqrt{16-52}}{2}$$

$$=\frac{4\pm\sqrt{-36}}{2}=\frac{4\pm 6i}{2}=2\pm 3i$$

28. (b) : $(5 - 3i)^3 = 125 - 225i - 135 + 27i = -10 - 198i$

29. (b) : $\arg\left(\frac{3+i}{2-i}+\frac{3-i}{2+i}\right)=\arg\left(\frac{6+5i+i^2+6-5i+i^2}{5}\right)$

$$=\arg\left(\frac{10}{5}\right)=\arg(2)=0$$

30. (a) : The given equality can be rewritten as

$(x^4 - 3x^2) + i(2x - y) = 4 + i(2y - 5)$

$\Rightarrow \quad x^4 - 3x^2 = 4,\ 2x - y = 2y - 5$

$\Rightarrow \quad (x^2 - 4)(x^2 + 1) = 0 \Rightarrow x = \pm 2 \qquad (\because x^2 \neq -1)$

$\therefore \quad$ At $x = 2$, $y = 3$ and at $x = -2$, $y = 1/3$

31. (d) : $(-i)(2i)\left(-\frac{1}{8}i\right)^3=2\times\frac{1}{8\times 8\times 8}\times i^5$

$$=\frac{1}{256}(i^2)^2\,i=\frac{1}{256}i=0+\frac{1}{256}i$$

32. (a) : Let $z = x + iy$ Then, $|z^2 - 1| = |z|^2 + 1$

$\Rightarrow \quad |x^2 - y^2 - 1 + i2xy| = |x + iy|^2 + 1$

$\Rightarrow \quad (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow \quad 4x^2 = 0 \Rightarrow x = 0$

Hence z lies on y-axis or imaginary axis.

33. (a) : Given, $x^2 + 3 = 0 \Rightarrow x^2 = -3$

$\Rightarrow \quad x=\pm\sqrt{-3}=\pm\sqrt{3}i$

34. (a) : $x=\frac{-(-14)\pm\sqrt{(-14)^2-4(1)(58)}}{2}$

$$=\frac{14\pm\sqrt{196-232}}{2}=\frac{14\pm 6i}{2}=7\pm 3i$$

35. (b) : Given, $5x^2 - 6x + 2 = 0$

$D = (-6)^2 - 4(5)(2) = 36 - 40 = -4$

$$x=\frac{-(-6)\pm\sqrt{-4}}{2(5)}=\frac{6\pm 2i}{10}=\frac{3\pm i}{5}$$

36. (a) : Given, $x^2 + 8 = 0 \Rightarrow x^2 = -8$

$\Rightarrow \quad x=\pm\sqrt{-8}=\pm 2\sqrt{2}i$

37. (c) : Given, $x^2 + 3x + 9 = 0$

$$x = \frac{-3 \pm \sqrt{9-4(9)}}{2} = \frac{-3 \pm \sqrt{-27}}{2} = \frac{-3 \pm 3\sqrt{3}i}{2}$$

38. (b) : We have, $4x + i(3x - y) = 3 + i(-6)$

Equating the real and imaginary parts, we get

$4x = 3$, $3x - y = -6$

On solving above questions, we get

$x = \frac{3}{4}$ and $y = \frac{33}{4}$.

39. (a) : Given $x = 2 + 5i$

$\Rightarrow \quad x - 2 = 5i$

$\Rightarrow \quad x^2 - 4x + 29 = 0$

Now, $x^3 - 5x^2 + 33x - 49$

$= x(x^2 - 4x + 29) - 1\,(x^2 - 4x + 29) - 20 = -20$

40. (b) : Since, $x^2 + y^2 = (x + iy)\,(x - iy)$

$\therefore \quad (p+iq) = \left(\frac{x^2+y^2}{x+iy}\right)i = (x-iy)i = y+ix$

$\therefore \quad p = y, q = x$

41. (c) : We have, $\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i}$

$$= \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-(\sqrt{2}i)^2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i$$

42. (d) : We have, $z = \frac{1+7i}{(2-i)^2} = \frac{1+7i}{(4+i^2-4i)}$

$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{3+4i+21i+28i^2}{9-16i^2} = \frac{3-28+25i}{25} = -1+i$

43. (d) : $(z-1)(\overline{z}-5)+(\overline{z}-1)(z-5)$

$= 2\,\mathrm{Re}[(z-1)(\overline{z}-5)] \qquad (\because z_1\overline{z}_2 + z_2\overline{z}_1 = 2\,\mathrm{Re}(z_1\overline{z}_2))$

$= 2\ \mathrm{Re}[(1 + i)(-3 - i)] = 2\,(-2) = -4 \quad$ (Given $z = 2 + i$)

44. (b) : $\frac{|1+i||2+i|}{|3+i|} = \frac{\sqrt{2}\sqrt{5}}{\sqrt{10}} = 1$

45. (d) : $|\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\overline{\beta} = 1$

$\therefore \quad \left|\frac{\beta-\alpha}{1-\overline{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta\overline{\beta}-\overline{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta(\overline{\beta}-\overline{\alpha})}\right|$

$$= \frac{|\beta-\alpha|}{|\beta||\overline{\beta}-\overline{\alpha}|} = \frac{|\beta-\alpha|}{1|\overline{\beta-\alpha}|} = \frac{|\beta-\alpha|}{|\beta-\alpha|} = 1$$

46. (a) : Since complex roots always occur in conjugate pair.

So, $2 - i$ is also a root of given equation.

Given equation is $x^3 - 5x^2 + 9x - 5 = 0$.

$x = 1$ satisfies this equation.

$\therefore \quad$ Other roots are $(2 - i)$ and 1.

47. (a) : Let α, β be the roots of the equation $x^2 - Px + 8 = 0$.

Then, $|\alpha - \beta| = 2$, $\alpha + \beta = P$ and $\alpha\beta = 8$.

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\therefore \quad (2)^2 = P^2 - 32 \Rightarrow P^2 - 32 = 4 \Rightarrow P = \pm 6$

48. (b) : Since complex roots always occur in conjugate pair.

$\therefore \quad$ Other root is $1 + i$.

Sum of roots $= \frac{-a}{1} = (1 - i) + (1 + i) \Rightarrow a = -2$

Product of roots $= \frac{b}{1} = (1 - i)(1 + i) \Rightarrow b = 2$

49. (d) : We have, $z^2 + |z|^2 = 0$

$\Rightarrow \quad x^2 - y^2 + i2xy + x^2 + y^2 = 0$ [Taking $z = x + iy$]

$\Rightarrow \quad 2x^2 + i2xy = 0$

$\Rightarrow \quad 2x^2 = 0$ and $2xy = 0$

[Equating real and imaginary part]

$\Rightarrow \quad x = 0$ and $xy = 0$

Thus, $x = 0$ and y can have any real value.

Hence infinitely many solutions.

50. (c) : Since complex roots always exist in conjugate pair, so other root will be $2 - 3i$.

Now, S = sum of roots $= 2 + 3i + 2 - 3i = 4$

and P = product of roots $= (2 + 3i)(2 - 3i) = 4 + 9 = 13$

So, required equation will be

$x^2 - Sx + P = 0 \Rightarrow x^2 - 4x + 13 = 0$

51. (d) : Clearly, $z = -3 + 4i$

$\therefore \quad |z| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

52. (c) : The multiplicative inverse of $-3 + 4i$ is

$$\frac{1}{-3+4i} = \frac{-1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{-(3+4i)}{25} = \frac{-3}{25} - \frac{4}{25}i$$

53. (c) : $\because \; z' = 1+\sqrt{3}i$

$\therefore \quad z' \cdot z = (1+\sqrt{3}i)(-3+4i) = 1(-3+4i)+\sqrt{3}i(-3+4i)$

$= -3+4i-3\sqrt{3}i+4\sqrt{3}i^2$

$= (-3-4\sqrt{3})+i\,(4-3\sqrt{3})$

54. (a) : $|z'| = \sqrt{1+3} = \sqrt{4} = 2$

55. (d) : $z + z' = -3 + 4i + 1 + \sqrt{3}i = -2 + (4 + \sqrt{3})\,i$

Thus, its conjugate is $-2 - (4 + \sqrt{3})\,i$

56. (c) : The required quadratic equation is given by

x^2 – (sum of roots) x + Product of roots = 0

$\Rightarrow \quad x^2 + 3x + 9 = 0$

57. (b) : Here, $a = 1$, $b = 3$, $c = 9$

Discriminant $= b^2 - 4ac$

$= 3^2 - 4 \cdot 1 \cdot 9 = 9 - 36 = -27 < 0$

58. (a) : $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow\ x=\dfrac{-3\pm\sqrt{-27}}{2}=\dfrac{-3\pm3\sqrt{3}\,i}{2}$

59. (a) : $|z_1z_2|$ = |Product of roots| = $|9| = 9$

60. (c) : Complex roots of quadratic equation are of the form $a\pm ib$.
Thus, occur in conjugate pair.

61. (b) : $|z_2z_3+8z_3z_1+27z_1z_2|=\left|z_1z_2z_3\left(\dfrac{1}{z_1}+\dfrac{8}{z_2}+\dfrac{27}{z_3}\right)\right|$

$=|z_1||z_2||z_3|\left|\dfrac{1}{z_1}+\dfrac{8}{z_2}+\dfrac{27}{z_3}\right|=|z_1||z_2||z_3|\left|\dfrac{\overline{z}_1}{|z_1|^2}+\dfrac{8\overline{z}_2}{|z_2|^2}+\dfrac{27\overline{z}_3}{|z_3|^2}\right|$

$=|z_1||z_2||z_3|\left|\dfrac{\overline{z}_1}{1}+\dfrac{8\overline{z}_2}{4}+\dfrac{27\overline{z}_3}{9}\right|=|z_1||z_2||z_3||\overline{z_1+2z_2+3z_3}|$

$=|z_1||z_2||z_3||z_1+2z_2+3z_3|=1\times2\times3\times6=36$

62. (b) : Since, $x=-2$ is a root of $f(x)=0$.

$\therefore\ f(x)=(x+2)(ax+b)$

But $f(0)+f(1)=0\Rightarrow 2b+3a+3b=0\Rightarrow -\dfrac{b}{a}=\dfrac{3}{5}$

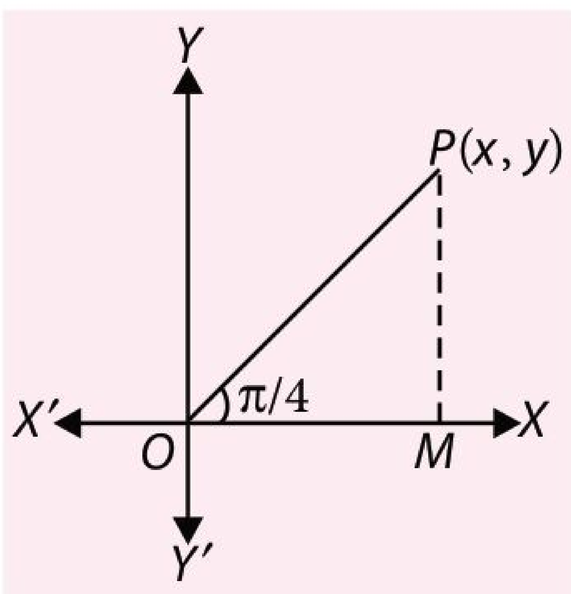
63. (c) : Statement-I is a standard result.
We have, $|z-(2+3i)|=4$

$\Rightarrow$ Distance of $P(z)$ from the point (2, 3) is equal to 4.

$\Rightarrow$ Locus of P is a circle with centre at (2, 3) and radius 4.

64. (c) : Let $z=x+iy$.

Since $\arg z=\dfrac{\pi}{4}$, therefore $\angle XOP=\dfrac{\pi}{4}$



$\Rightarrow\ \tan(\angle XOP)=1$

$\Rightarrow\ MP=OM\Rightarrow y=x\neq0\ (\because\ z\neq0)$

Hence $z^2=(x+yi)^2=(x+xi)^2$

$=x^2+i^2x^2+2x^2i$

$=x^2-x^2+2x^2i=2x^2i$

$\Rightarrow\ \mathrm{Re}\,(z^2)=0$

We have seen that if $\arg z=\dfrac{\pi}{4}$ and $z\neq0$ then $\mathrm{Re}\,z=\mathrm{Im}z$.

65. (a) : We have, $3x^2-3x+2=0$

Here, $b^2-4ac=9-4(3)(2)=9-24=-15<0$

$\therefore$ Roots are non-real.

66. (a) : We have, $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}=\dfrac{2-5i}{1-4i}=\dfrac{2-5i}{1-4i}\times\dfrac{1+4i}{1+4i}$

$=\dfrac{(2+20)+i(8-5)}{1-16i^2}=\dfrac{22+3i}{17}=\dfrac{22}{17}+\dfrac{3}{17}i$

Also, Statement-II is a standard result.

67. (d) : Clearly statement-II is true.

Let $z=\dfrac{(1+i)^2}{3-i}=\dfrac{1+2i+i^2}{3-i}=\dfrac{2i}{3-i}\times\dfrac{3+i}{3+i}$

$=\dfrac{6i+2i^2}{9-i^2}=\dfrac{6i-2}{10}=-\dfrac{1}{5}+\dfrac{3}{5}i\ \therefore\ \overline{z}=-\dfrac{1}{5}-\dfrac{3}{5}i$

68. (b) : $\tan\alpha=\left|\dfrac{2\sqrt{3}}{-2}\right|=\sqrt{3}=\tan\dfrac{\pi}{3}\Rightarrow\alpha=\dfrac{\pi}{3}$

Clearly, $\mathrm{Re}(z)<0$ and $\mathrm{Im}\,(z)>0$. So, the point representing z lies in the second quadrant.

$\therefore\ \arg(z)=\pi-\alpha=\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$.

69. (d) : Let $z=x+iy$. Then, $z-2-3i=(x+iy)-2-3i$
$=(x-2)+i(y-3)$

Now, $\tan\dfrac{\pi}{4}=\dfrac{y-3}{x-2}\Rightarrow1=\dfrac{y-3}{x-2}$

$\Rightarrow\ x-y+1=0$, which is a straight line.

70. (d) : Let $z=\sqrt{37}+\sqrt{-19}=\sqrt{37}+\sqrt{19\times(-1)}$

$=\sqrt{37}+\sqrt{19}\sqrt{-1}=\sqrt{37}+i\sqrt{19}$

$\therefore\ \mathrm{Re}(z)=\sqrt{37}$ and $\mathrm{Im}(z)=\sqrt{19}$

MONTHLY TEST DRIVE

XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 3 : Trigonometric Function

Time Taken : 60 Min.

Only One Option Correct Type

1. In a ΔABC if $(\sqrt{3}-1)a = 2b,\ A = 3B,$ then C is

(a) 60° (b) 120°

(c) 30° (d) 45°

2. If $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$, then the general value of θ is

(a) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}$ (b) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{6}$

(c) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{3}$ (d) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{6}$

3. The value of $\cot 16° \times \cot 44° + \cot 44° \times \cot 76° - \cot 76° \times \cot 16°$ is

(a) 4 (b) 1

(c) 3 (d) 0

4. If $\tan^2\alpha\ \tan^2\beta + \tan^2\beta\ \tan^2\gamma + \tan^2\gamma\ \tan^2\alpha + 2\tan^2\alpha\ \tan^2\beta\ \tan^2\gamma = 1$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

(a) 0 (b) –1

(c) 1 (d) None of these

5. The solution of the equation $\log_{\cos x}\sin x + \log_{\sin x}\cos x = 2$ is given by

(a) $x = 2n\pi + \frac{\pi}{4}, n \in I$ (b) $x = n\pi + \frac{\pi}{2}, n \in I$

(c) $x = n\pi + \frac{\pi}{8}, n \in I$ (d) None of these

6. In any triangle, with usual notations the minimum value of $r_1r_2r_3/r^3$ is equal to

(a) 1 (b) 9

(c) 27 (d) None of these

One or More Than One Option(s) Correct Type

7. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then which of the following is/are true?

(a) $\cos(A - B) = \frac{1}{3}$

(b) $|\cos A - \cos B| = \sqrt{\frac{2}{3}}$

(c) $\cos(A - B) = -\frac{1}{3}$

(d) $|\cos A - \cos B| = \frac{1}{2\sqrt{3}}$

8. If $\sin(\alpha + \beta) = 1$ and $\sin(\alpha - \beta) = 1/2$, where $\alpha, \beta \in [0, \pi/2]$, then

(a) $\tan(\alpha + 2\beta) = -\sqrt{3}$

(b) $\tan(2\alpha + \beta) = -1/\sqrt{3}$

(c) $\tan(\alpha + 2\beta) = \sqrt{3}$

(d) $\tan(2\alpha + \beta) = 1/\sqrt{3}$

9. If sides of triangle ABC are a, b and c such that $2b = a + c$, then

(a) $\frac{b}{c} > \frac{2}{3}$ (b) $\frac{b}{c} > \frac{1}{3}$

(c) $\frac{b}{c} < 2$ (d) $\frac{b}{c} < \frac{1}{2}$

10. If $\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x = 7$ and $\sin 2x = a - b\sqrt{7}$, then

(a) $a = 8$ (b) $b = 22$

(c) $a = 22$ (d) $b = 8$

11. If $\cos(\theta - \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in H.P., then $\cos\theta \sec\frac{\alpha}{2}$ is equal to

(a) -1 (b) $-\sqrt{2}$

(c) $\sqrt{2}$ (d) 2

12. If A is the area and $2s$ is the sum of the sides of a triangle, then

(a) $A \le \frac{s^2}{4}$ (b) $A \le \frac{s^2}{3\sqrt{3}}$

(c) $A < \frac{s^2}{\sqrt{3}}$ (d) None of these

13. In a ΔABC,

(a) $\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}$

(b) $\sin^2 A + \sin^2 B + \sin^2 C \le \frac{9}{4}$

(c) $\sin A \sin B \sin C$ is always positive

(d) $\sin^2 A + \sin^2 B \le 1 + \cos C$

Comprehension Type

Paragraph for Q. No. 14 and 15

The sides of a triangle ABC are 7, 8, 6 the smallest angle being 'C'.

14. The length of the median from vertex C is

(a) $\frac{\sqrt{95}}{4}$ (b) $\frac{\sqrt{95}}{2}$

(c) $\sqrt{\frac{95}{2}}$ (d) $\frac{\sqrt{95}}{3}$

15. The length of the internal bisector of angle C is

(a) $\sqrt{30}$ (b) $\frac{14}{5}\sqrt{6}$

(c) $\frac{14}{5}$ (d) $2\sqrt{6}$

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	Max $\{5\sin\theta + 3\sin(\theta - \alpha), \theta \in R\} = 7$, then the set of possible values of α is	1.	$2n\pi + 3\pi/4, n \in Z$
Q.	If $x \ne \frac{(2n+1)\pi}{2}$ and $(\cos x)^{(\sin^2 x - 3\sin x + 2)} = 1$, then $x =$	2.	$2n\pi \pm \frac{\pi}{3}, n \in Z$
R.	If $\sqrt{(\sin x)} + 2^{1/4}\cos x = 0$, then $x =$	3.	$2n\pi + \cos^{-1}(1/3), n \in Z$
S.	If $\log_5 \tan x = (\log_5 4)(\log_4(3\sin x))$, then $x =$	4.	$2n\pi, n \in Z$

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	1	3
(c)	4	2	3	1
(d)	4	1	2	3

Numerical Answer Type

17. If $\frac{1}{\cos 290^\circ} + \frac{1}{\sqrt{3}\sin 250^\circ} = \lambda$, then the value of $9\lambda^4 + 81\lambda^2 + 97$ must be ________.

18. Consider a ΔABC, in which the sides are $a = (n + 1)$, $b = (n + 2)$, $c = n$ with $\tan C = 4/3$, then the value of $\Delta/12$ is ________.

19. If $3\sin x + 4\cos x = 5$, then the value of $90\tan^2(x/2) - 60\tan(x/2) + 16$ is equal to ________.

20. If $k\left[\tan^6\frac{\pi}{9} - 33\tan^4\frac{\pi}{9} + 27\tan^2\frac{\pi}{9}\right] = 9$, then the value of k is ________.

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CBSE warm-up!

CLASS-XII

TERM-I OBJECTIVE TYPE QUESTIONS*

Series 2 Inverse Trigonometric Functions

MCQs

1. Find the principal value of $\sec^{-1}(2)$.

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) 0

2. $\sin^{-1}\left(\frac{-1}{2}\right) =$

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

3. Evaluate : $\cos\left\{-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}$

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

4. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=$

(a) $\frac{\pi}{2}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{6}$

5. If $\theta = \tan^{-1}a$, $\phi = \tan^{-1} b$ and $ab = -1$, then $|\theta - \phi|$ is equal to

(a) 0 (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) None of these

6. Domain of $\cos^{-1}[x]$ (where $[\cdot]$ denotes G.I.F.) is

(a) $[-1, 2]$ (b) $[-1, 2)$
(c) $(-1, 2]$ (d) None of these

7. Evaluate : $\operatorname{cosec}\left\{\cot^{-1}\left(\frac{4}{3}\right)\right\}$

(a) $\frac{3}{4}$ (b) $\frac{5}{3}$ (c) $\frac{3}{5}$ (d) $\frac{4}{3}$

8. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)+4\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{3\pi}{4}$

9. Evaluate :
$\cot^{-1}(1) + \operatorname{cosec}^{-1}(\sqrt{2}) + \sec^{-1}(2)$

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

10. Find the value of $\cot\left[\sin^{-1}\left\{\cos(\tan^{-1}1)\right\}\right]$.

(a) 0 (b) $\pi/4$
(c) 1 (d) does not exist.

11. Range of $f(x) = \cos^{-1} x + \sec^{-1}x$ is

(a) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ (b) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

(c) $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right]$ (d) None of these

12. Evaluate : $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{4}$ (d) $-\frac{\pi}{2}$

13. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

14. Find the domain of $\sec^{-1}(2x + 1)$.

(a) $(-\infty, -1]$ (b) $[0, \infty)$
(c) $R-(-1, 1)$ (d) None of these

**Chapterwise practice questions for CBSE Exam Term- I as per the pattern issued by CBSE.*

15. $\tan^{-1}(-\sqrt{3})+\sec^{-1}(-2)-\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is equal to

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) 0

16. If $x, y, z \in [-1, 1]$ such that $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=0$, then find the value of $x + y + z$.

(a) 0 (b) 1 (c) 2 (d) 3

17. $\cos^{-1}\left(\frac{-1}{2}\right)+2\sin^{-1}\left(\frac{-1}{2}\right)$ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{8}$

18. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$ equals

(a) 0 (b) 1 (c) 6 (d) 12

19. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ equals to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$ (d) $\frac{2\pi}{3}$

20. If $\cot^{-1}\left(-\frac{1}{5}\right) = x$, then the value of $\sin x$ is

(a) $\frac{1}{\sqrt{26}}$ (b) $\frac{-1}{\sqrt{26}}$ (c) $\frac{5}{\sqrt{26}}$ (d) $\frac{-5}{\sqrt{26}}$

21. The principal value of $\cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{6}$ (d) none of these

22. The domain of $\operatorname{cosec}^{-1}(3 - 2x)$ is

(a) $(-\infty, -1]$ (b) $[2, \infty)$

(c) $R - (-1, 1)$ (d) None of these

23. Evaluate : $\sin^{-1}(\sin(-600°))$.

(a) 0 (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{-\pi}{3}$

24. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then $x^2 + y^2 + z^2 + 2xyz =$

(a) 0 (b) 1

(c) -1 (d) None of these

25. Evaluate : $\sin\left(2\cot^{-1}\left(-\frac{5}{12}\right)\right)$.

(a) $\frac{1}{169}$ (b) $\frac{-1}{169}$ (c) $\frac{120}{169}$ (d) $\frac{-120}{169}$

26. $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos(\tan^{-1}2\sqrt{2}) =$

(a) $\frac{13}{15}$ (b) $\frac{15}{13}$ (c) $\frac{14}{15}$ (d) $\frac{15}{14}$

27. If $\tan^{-1}\frac{4}{3}=\theta$, find the value of $\cos\theta$.

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{5}{3}$ (d) $\frac{5}{4}$

28. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{-\pi}{3}$

29. Evaluate :

$$\cot^{-1}(-\sqrt{3})+\tan^{-1}(1)+\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

(a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

30. The value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

31. The value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{3}$

32. Find the principal value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

33. The value of $\cos^2\left(\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\right)$ is

(a) $\frac{4}{5}$ (b) $\frac{5}{4}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

34. The value of $\sin\left(\cot^{-1}\frac{3}{4}\right)$ is

(a) $\frac{4}{5}$ (b) $\frac{5}{4}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

35. The value of $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right) + \tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right)$ is

(a) $\frac{a}{b}$ (b) $\frac{2b}{a}$ (c) $\frac{b}{a}$ (d) $\frac{b}{2a}$

36. The number of triplets (x, y, z) satisfies the equation $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ is

(a) 1 (b) 2 (c) 0 (d) infinite

37. If $6\sin^{-1}(x^2 - 6x + 8.5) = \pi$, then x is equal to
(a) 1 (b) 2 (c) 3 (d) 8

38. If $\sin^{-1}(x^2 - 7x + 12) = 0$, then $x =$
(a) -2 (b) 4 (c) -3 (d) 5

39. If $\tan^{-1}(x^2 - 4x + 4) = 0$, then x equals to
(a) 2 (b) 4 (c) 3 (d) 5

40. If $\tan^{-1}(\cot\theta) = 2\theta$, then θ is equal to
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) None of these

41. Which of the following is the principal value branch of $\sec^{-1}x$?
(a) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (b) $(0, \pi)$
(c) $[0, \pi]$ (d) $[0, \pi]-\left\{\frac{\pi}{2}\right\}$

42. Evaluate : $\cos\left(2\cos^{-1}\left(\frac{2}{5}\right)\right)$
(a) $\frac{13}{25}$ (b) $\frac{17}{25}$ (c) $\frac{-13}{25}$ (d) $\frac{-17}{25}$

43. Evaluate : $\sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$.
(a) $\sqrt{3}/2$ (b) 1/2 (c) 0 (d) 1

44. $\cot^{-1}(-\sqrt{3})$ is equal to
(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

45. The value of $\tan^{-1}(1) + \tan^{-1}(0) + \tan^{-1}(-1)$ is equal to
(a) π (b) $\frac{5\pi}{4}$
(c) $\frac{\pi}{2}$ (d) None of these

46. The solution set of the equation $\tan^{-1} x - \cot^{-1}x = \cos^{-1}(2 - x)$ is
(a) $[0, 1]$ (b) $[-1, 1]$
(c) $[1, 3]$ (d) None of these

47. $\cos^{-1}[\cos(2\cot^{-1}(\sqrt{3}))] =$
(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

48. If $2(\sin^{-1} x)^2 - 5(\sin^{-1} x) + 2 = 0$, then $x =$
(a) $\pi/6$ (b) $\pi/3$ (c) 2 (d) 1/2

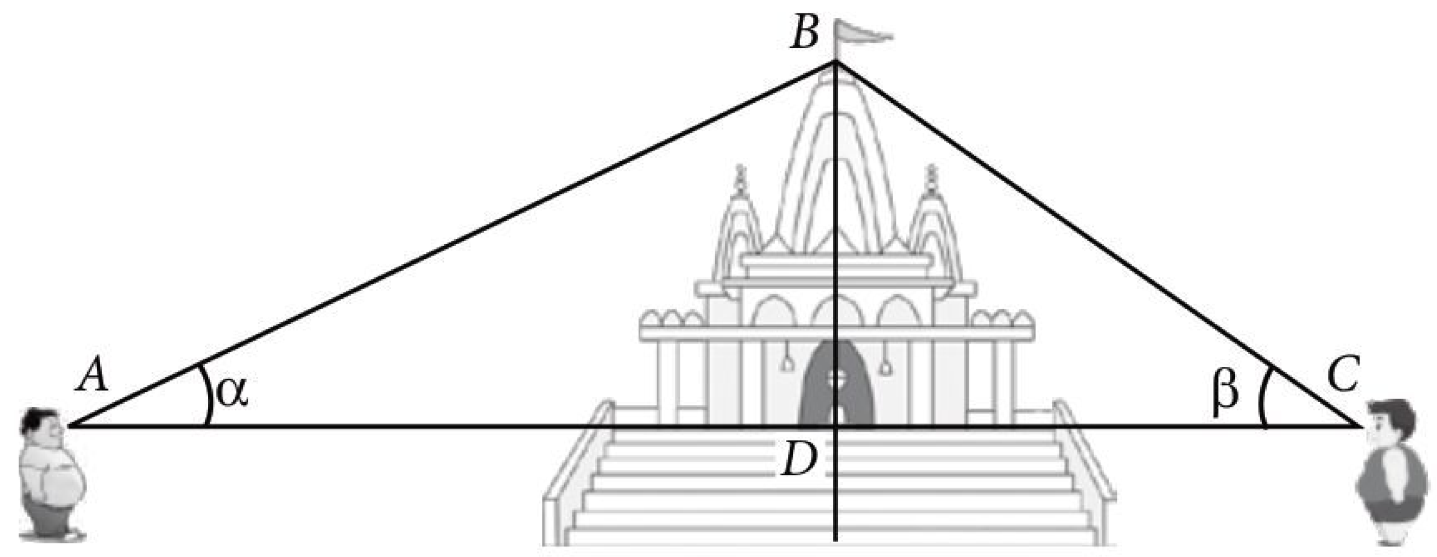
49. The principal solution of $\cos^{-1}\left(\cos\left(\frac{9\pi}{4}\right)\right)$ is
(a) $\frac{7\pi}{4}$ (b) $\frac{-\pi}{4}$ (c) $\frac{9\pi}{4}$ (d) $\frac{\pi}{4}$

50. The principal solution of $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right)$ is
(a) $\frac{4\pi}{3}$ (b) $\frac{5\pi}{3}$ (c) $\frac{-5\pi}{3}$ (d) $\frac{-\pi}{3}$

CASE BASED

Case I : Read the following and answer any four questions from 51 to 55 given below.

Two men on either side of a temple, which is 30 metres high above the stairs, observe its top at the angles of elevation α and β respectively (as shown in the figure below). The distance between the two men is $40\sqrt{3}$ metres and the distance between the first person A and the temple is $30\sqrt{3}$ metres.



51. $\angle CAB = \alpha =$
(a) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\sin^{-1}\left(\frac{1}{2}\right)$
(c) $\sin^{-1}(2)$ (d) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

52. $\angle CAB = \alpha =$
(a) $\cos^{-1}\left(\frac{1}{5}\right)$ (b) $\cos^{-1}\left(\frac{2}{5}\right)$
(c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\cos^{-1}\left(\frac{4}{5}\right)$

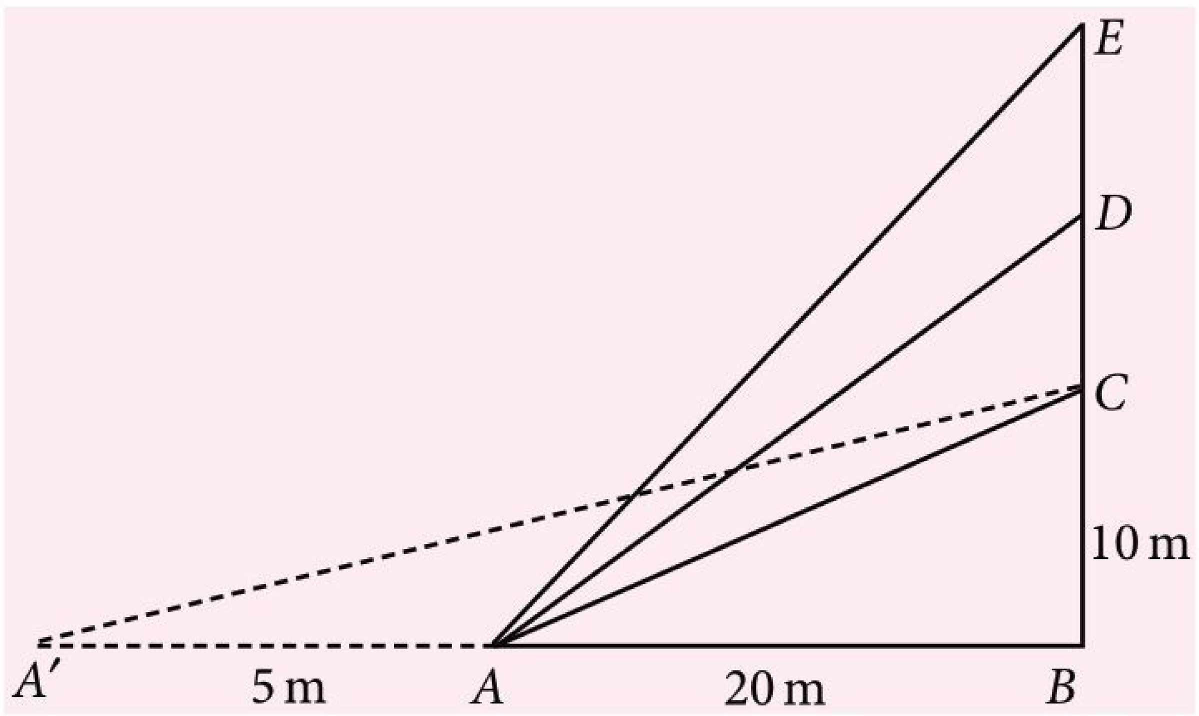
53. $\angle BCA = \beta =$
(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}(2)$
(c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\tan^{-1}(\sqrt{3})$

54. $\angle ABC =$
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

55. Domain and range of $\cos^{-1} x$ are respectively
(a) $(-1, 1), (0, \pi)$ (b) $[-1, 1], (0, \pi)$
(c) $[-1, 1], [0, \pi]$ (d) $(-1, 1), \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Case II : Read the following and answer any four questions from 56 to 60 given below.

The Government of India is planning to fix a hoarding board at the face of a building on the road of a busy market for awareness of Covid-19 protocol. Ram, Robert and Rahim are the three engineers who are working on this project. "*A*" is considered as a person, viewing the hoarding board, 20 metres away from the building and standing at the edge of a pathway nearby. Ram, Robert and Rahim suggested to the firm to place the hoarding board at three different locations namely, *C, D* and *E*. "*C*" is at the height of 10 metres from the ground level. For the viewer *A*, the angle of elevation of "*D*" is double the angle of elevation of "*C*". The angle of elevation of "*E*" is triple the angle of elevation of "*C*" for the same viewer.



56. Measure of $\angle CAB$ =
(a) $\tan^{-1}(2)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$
(c) $\tan^{-1}(1)$ (d) $\tan^{-1}(3)$

57. Measure of $\angle DAB$ =
(a) $\tan^{-1}\left(\frac{3}{4}\right)$ (b) $\tan^{-1}(3)$
(c) $2\tan^{-1}\left(\frac{1}{2}\right)$ (d) $\tan^{-1}(4)$

58. Measure of $\angle EAB$ =
(a) $\tan^{-1}(11)$ (b) $\tan^{-1}3$
(c) $\tan^{-1}\left(\frac{2}{11}\right)$ (d) $3\tan^{-1}\left(\frac{1}{2}\right)$

59. If A' is another viewer, which is at a distance of 5 m from A as shown in figure, then the difference between $\angle CAB$ and $\angle CA'B$ is
(a) $\tan^{-1}(1/2)$ (b) $\tan^{-1}(1/2) - \tan^{-1}(2/5)$
(c) $\tan^{-1}\left(\frac{2}{5}\right)$ (d) $\tan^{-1}\left(\frac{11}{21}\right)$

60. Domain and range of $\tan^{-1} x$ are respectively
(a) $R^+, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $R^-, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $R, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $R, \left(0, \frac{\pi}{2}\right)$

ASSERTION & REASON

Directions (Q.No. 61-70) : In the following questions, a statement of assertion (Statement-I) is followed by a statement of reason (Statement-II). Mark the correct choice as :

(a) If both Statement-I and Statement-II are true and Statement-II is the correct explanation of Statement-I.
(b) If both Statement-I and Statement-II are true but Statement-II is not the correct explanation of Statement-I.
(c) If Statement-I is true but Statement-II is false.
(d) If Statement-I is false but Statement-II is true.

61. **Statement-I :** Principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is $\frac{\pi}{3}$.
Statement-II : Principal value branch of arc sin function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

62. **Statement-I :** Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.
Statement-II : $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is defined for all $x \in [-1, 1]$.

63. **Statement-I :** Number of roots of the equation $\cot^{-1}x + \cos^{-1}2x + \pi = 0$ is zero.
Statement-II : Range of $\cot^{-1}x$ and $\cos^{-1}x$ is $(0, \pi)$ and $[0, \pi]$, respectively.

64. **Statement-I :** Range of $f(x) = \cot^{-1}(2x - x^2)$ is $(0, \pi)$.
Statement-II : $\cot^{-1}x$ is defined for all $x \in R$.

65. **Statement-I :** The domain for $f(x) = \sin^{-1}(1 + x^2)$ is $\{0, 1\}$.
Statement-II : $\sin^{-1}x$ is defined only if $x \in [-1, 1]$.

66. **Statement-I :** The value of $\sin(2\tan^{-1}(0.75)$, is 0.96.
Statement-II : Range of $\sin^{-1}(x)$ is $[0, \pi]$.

67. **Statement-I :** The value of $\cos\left(\frac{\pi}{3}-\cos^{-1}\frac{1}{2}\right)$ is 0.

Statement-II : If $y = \cos x$, then $x = \cos^{-1}y$, where $x \in [0, \pi]$.

68. **Statement-I :** The principal solution of $4\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$.

Statement-II : $\tan^{-1} x$ is defined for all $x \in R$.

69. **Statement-I :** The domain of function defined by $f(x)=\sin^{-1}\sqrt{x-1}$ is [1, 2].

Statement-II : $\sin^{-1}x$ is defined for all $x \in R$.

70. **Statement-I :** The domain of $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is R.

Statement-II : Dom $(f + g)(x)$ = Dom $f(x) \cap$ Dom $g(x)$

SOLUTIONS

1. **(c) :** Let $\sec^{-1}(2) = \theta \Rightarrow \sec\theta = 2 = \sec\frac{\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\therefore$ Principal value of $\sec^{-1}(2)$ is $\frac{\pi}{3}$.

2. **(d) :** Let $\sin^{-1}\left(\frac{-1}{2}\right)=\theta \Rightarrow \sin\theta=\frac{-1}{2}=-\sin\frac{\pi}{6}=\sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow \theta=\frac{-\pi}{6}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$\therefore$ Principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is $\left(\frac{-\pi}{6}\right)$.

3. **(b) :** $\cos\left[-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=\cos\left[-\frac{\pi}{6}+\frac{\pi}{6}\right]$

$=\cos 0 = 1$

4. **(d) :** Let $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$

$\Rightarrow \cos\theta=-\frac{\sqrt{3}}{2}=-\cos\frac{\pi}{6}=\cos\left(\pi-\frac{\pi}{6}\right)=\cos\frac{5\pi}{6}$

$\Rightarrow \theta=\frac{5\pi}{6}\in[0,\pi] \quad \therefore \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$

5. **(c) :** Given, $\theta = \tan^{-1} a$ and $\phi = \tan^{-1} b$ and $ab = -1$

$\therefore$ $\tan\theta\tan\phi = ab = -1$

$\Rightarrow$ $\tan\theta = -\cot\phi$ or $\tan\phi = -\cot\theta$

$\Rightarrow \tan\theta=\tan\left(\frac{\pi}{2}+\phi\right)$ or $\tan\phi = \tan\left(\frac{\pi}{2}+\theta\right)$

$\Rightarrow \theta-\phi=\frac{\pi}{2}$ or $\phi-\theta=\frac{\pi}{2}$

$\Rightarrow |\theta-\phi|=\frac{\pi}{2}$

6. **(b) :** Clearly, $-1 \le [x] \le 1$

$\Rightarrow -1 \le x < 2 \Rightarrow x \in [-1, 2)$

7. **(b) :** Let $\theta = \cot^{-1}\left(\frac{4}{3}\right) \Rightarrow \cot\theta=\frac{4}{3}$

Now, $\text{cosec}\left\{\cot^{-1}\left(\frac{4}{3}\right)\right\}=\text{cosec}\,\theta=\sqrt{1+\cot^2\theta}$

$=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$

8. **(c) :** $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)+4\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$=\frac{\pi}{3}+2\cdot\frac{\pi}{6}+4\cdot\frac{\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}+\frac{2\pi}{3}=\frac{4\pi}{3}$

9. **(b) :** $\cot^{-1}(1)+\text{cosec}^{-1}(\sqrt{2})+\sec^{-1}(2)$

$=\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{3}=\frac{5\pi}{6}$

10. **(c) :** We have, $\cot[\sin^{-1}\{\cos(\tan^{-1} 1)\}]$

$=\cot\left\{\sin^{-1}\left(\cos\frac{\pi}{4}\right)\right\}=\cot\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)=\cot\frac{\pi}{4}=1$

11. **(d) :** Given $f(x) = \cos^{-1}x + \sec^{-1}x$

Domain of $\cos^{-1}x = [-1, 1]$

Domain of $\sec^{-1}x = (-\infty, \infty) - (-1, 1)$

Domain of $f(x) = [-1, 1] \cap [(-\infty, \infty) - (-1, 1)] = \{-1, 1\}$

Now, $f(-1) = \cos^{-1}(-1) + \sec^{-1}(-1) = \pi + \pi = 2\pi$

and $f(1) = \cos^{-1}(1) + \sec^{-1}(1) = 0 + 0 = 0$

$\therefore$ Range of $f(x)$ is $\{0, 2\pi\}$.

12. **(c) :** $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{4}\right)\right\}$

$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\tan^{-1}\left\{\tan\left(-\frac{\pi}{4}\right)\right\}=-\frac{\pi}{4}$

13. **(d) :** The value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2\times\frac{\pi}{6}$

$=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

14. **(d) :** $\sec^{-1}(2x + 1)$ is defined if

$2x + 1 \ge 1$ or $2x + 1 \le -1$

$\Rightarrow$ $2x \ge 0$ or $2x \le -2$ $\Rightarrow$ $x \ge 0$ or $x \le -1$

$\Rightarrow$ $x \in (-\infty, -1] \cup [0, \infty)$

Hence, the domain of $\sec^{-1}(2x + 1)$ is $(-\infty, -1] \cup [0, \infty)$

15. (d) : The value of

$$\tan^{-1}(-\sqrt{3})+\sec^{-1}(-2)-\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)=\frac{-\pi}{3}+\frac{2\pi}{3}-\frac{\pi}{3}=0$$

16. (d) : We have, $x, y, z \in [-1, 1]$

$\Rightarrow \quad -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1$

$\Rightarrow \quad 0 \le \cos^{-1} x \le \pi, 0 \le \cos^{-1} y \le \pi, 0 \le \cos^{-1} z \le \pi$

$\because \quad \cos^{-1}x + \cos^{-1}y + \cos^{-1} z = 0$ [Given]

$\Rightarrow \quad \cos^{-1} x = 0, \cos^{-1} y = 0$ and $\cos^{-1}z = 0$

$\Rightarrow \quad x = y = z = 1$. Hence, $x + y + z = 3$.

17. (a) : Principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is $\frac{2\pi}{3}$

and principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is $\left(\frac{-\pi}{6}\right)$.

$$\therefore \quad \cos^{-1}\left(\frac{-1}{2}\right)+2\sin^{-1}\left(\frac{-1}{2}\right)$$

$$=\frac{2\pi}{3}+\left(2\times\frac{-\pi}{6}\right)=\frac{2\pi}{3}-\frac{\pi}{3}=\frac{\pi}{3}$$

18. (c) : $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

$\because \quad 0 \le \cos^{-1}x \le \pi$

$\Rightarrow \quad \cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi \Rightarrow \alpha = \beta = \gamma = -1$

$\therefore \quad \alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$

$= -1(-1-1) + (-1)(-1-1) + (-1)(-1-1)$

$= 2 + 2 + 2 = 6$

19. (c) : Let $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)=\theta$

$$\Rightarrow \sec\theta=\frac{-2}{\sqrt{3}}=-\sec\left(\frac{\pi}{6}\right)=\sec\left(\pi-\frac{\pi}{6}\right)=\sec\frac{5\pi}{6}$$

$$\Rightarrow \theta=\frac{5\pi}{6}\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$$

$\therefore \quad$ Principal value of $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ is $\frac{5\pi}{6}$.

20. (c) : Given, $\cot^{-1}\left(-\frac{1}{5}\right)=x$

$$\Rightarrow \cot x=-\frac{1}{5}, 0<x<\pi$$

$$\therefore \operatorname{cosec} x=\sqrt{1+\cot^2 x}=\sqrt{1+\left(-\frac{1}{5}\right)^2}=\frac{\sqrt{26}}{5}$$

[$\because$ cosec x is positive]

$$\Rightarrow \sin x=\frac{5}{\sqrt{26}}$$

21. (c) : We know that $\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$.

$$\therefore \quad \cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\}=\cos^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6} \qquad \left[\because \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}\right]$$

22. (d) : For the domain of $\operatorname{cosec}^{-1}(3-2x)$, $|3-2x| \ge 1$

$\Rightarrow \quad 3 - 2x \le -1$ or $3 - 2x \ge 1$

$\Rightarrow \quad -2x \le -4$ or $-2x \ge -2 \Rightarrow x \ge 2$ or $x \le 1$.

$\therefore \quad$ Domain of $\operatorname{cosec}^{-1}(3-2x)$ is $(-\infty, 1] \cup [2, \infty)$.

23. (b) : $\sin^{-1}(\sin(-600°))=\sin^{-1}\left\{\sin\left(-600\times\frac{\pi}{180}\right)\right\}$

$$=\sin^{-1}\left\{\sin\left(-\frac{10\pi}{3}\right)\right\}=\sin^{-1}\left(-\sin\frac{10\pi}{3}\right)$$

$$=\sin^{-1}\left\{-\sin\left(3\pi+\frac{\pi}{3}\right)\right\}=\sin^{-1}\left\{-\left(-\sin\frac{\pi}{3}\right)\right\}$$

$$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$$

24. (b) : Let $\cos^{-1} x = \alpha$, $\cos^{-1} y = \beta$ and $\cos^{-1} z = \gamma$

$\Rightarrow \quad \cos\alpha = x, \cos\beta = y$ and $\cos\gamma = z$

Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \quad \alpha + \beta + \gamma = \pi \;\therefore\; \alpha + \beta = \pi - \gamma$

$\therefore \quad \cos(\alpha + \beta) = \cos(\pi - \gamma)$

$\Rightarrow \quad \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\cos\gamma$

$\Rightarrow \quad xy-\sqrt{1-x^2}\sqrt{1-y^2}=-z$

$\Rightarrow \quad xy+z=\sqrt{1-x^2}\sqrt{1-y^2}$

Squaring both sides, we get

$x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$.

25. (d) : Let $\cot^{-1}\left(-\frac{5}{12}\right)=x, 0<x<\pi$

$$\Rightarrow \quad \cot x=-\frac{5}{12}, 0<x<\pi.$$

For these values of x, cosec $x > 0$.

MONTHLY TEST DRIVE CLASS XII ANSWER KEY

1.	(b)	**2.**	(c)	**3.**	(d)	**4.**	(a)	**5.**	(d)
6.	(d)	**7.**	(a,b,c)	**8.**	(b,c)	**9.**	(b,c,d)	**10.**	(b,c,d)
11.	(a,c,d)	**12.**	(c)	**13.**	(a,c,d)	**14.**	(a)	**15.**	(b)
16.	(a)	**17.**	(144)	**18.**	(0)	**19.**	(2)	**20.**	(4)

$\therefore \quad \operatorname{cosec} x=\sqrt{1+\cot^2 x}=\sqrt{1+\left(-\frac{5}{12}\right)^2}=\frac{13}{12}$

$\Rightarrow \quad \sin x=\frac{12}{13}.$

Now, $\cos x=\frac{\cos x}{\sin x}\cdot\sin x=\cot x\cdot\sin x$

$=\left(-\frac{5}{12}\right)\cdot\frac{12}{13}=-\frac{5}{13}.$

$\therefore \quad \sin\left(2\cot^{-1}\left(-\frac{5}{12}\right)\right)=\sin 2x=2\sin x\cos x=-\frac{120}{169}.$

26. (c) : Let $\tan^{-1}\frac{1}{3}=\alpha\Rightarrow\tan\alpha=\frac{1}{3}.$

$\sin\left(2\tan^{-1}\frac{1}{3}\right)=\sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha}=\frac{2\times\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}=\frac{3}{5}$

Let $\tan^{-1}2\sqrt{2}=\beta\Rightarrow\tan\beta=2\sqrt{2}.$

$\sec\beta=\sqrt{1+\tan^2\beta}=\sqrt{1+\left(2\sqrt{2}\right)^2}=3\Rightarrow\cos\beta=\frac{1}{3}.$

$\Rightarrow \quad \cos\left(\tan^{-1}2\sqrt{2}\right)=\cos\beta=\frac{1}{3}.$

$\therefore \quad \sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)=\frac{3}{5}+\frac{1}{3}=\frac{14}{15}.$

27. (a) : Given, $\tan^{-1}\frac{4}{3}=\theta$, where $\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

$\therefore \quad \tan\theta=\frac{4}{3}.$

We know that $\cos\theta>0$, when $\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right).$

$\therefore \quad \cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{1+\tan^2\theta}}=\frac{1}{\sqrt{1+\frac{16}{9}}}=\frac{3}{5}$

28. (c) : Since, $\sin\frac{2\pi}{3}=\sin\left(\pi-\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$\therefore \quad \sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}.$

Hence, the principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is $\frac{\pi}{3}.$

29. (b) : $\cot^{-1}(-\sqrt{3})+\tan^{-1}(1)+\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$=\left(\pi-\frac{\pi}{6}\right)+\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{6}+\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{4}$

30. (c) : $\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{6}\right)\right)$

$=\tan^{-1}\left(-\tan\left(\frac{\pi}{6}\right)\right)=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

31. (c) : $\cos^{-1}\cos\left(2\pi+\frac{\pi}{6}\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

$=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

32. (c) : $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left\{\tan\left(\pi+\frac{\pi}{6}\right)\right\}$

$=\tan^{-1}\left\{\tan\left(\frac{\pi}{6}\right)\right\}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

33. (a) : Let $\cos^{-1}\frac{3}{5}=\theta \Rightarrow \cos\theta=\frac{3}{5}$

$\therefore \quad \cos^2\left\{\left(\frac{1}{2}\right)\cos^{-1}\left(\frac{3}{5}\right)\right\}=\cos^2\left(\frac{\theta}{2}\right)$

$=\frac{\cos\theta+1}{2}=\frac{\frac{3}{5}+1}{2}=\frac{4}{5}$

34. (a) : Let $\cot^{-1}\frac{3}{4}=\theta \Rightarrow \cot\theta=\frac{3}{4}$

Now, $\operatorname{cosec}\theta=\sqrt{1+\cot^2\theta}=\sqrt{1+\frac{9}{16}}=\frac{5}{4}$

$\therefore \quad \sin\theta=\frac{4}{5} \quad \therefore \quad \sin\left(\cot^{-1}\left(\frac{3}{4}\right)\right)=\sin\theta=\frac{4}{5}$

35. (b) : Let $\cos^{-1}\frac{a}{b}=\theta \Rightarrow \cos\theta=\frac{a}{b}$

So, we have, $\tan\left(\frac{\pi}{4}+\frac{1}{2}\theta\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\theta\right)$

$=\frac{1+\tan\theta/2}{1-\tan\theta/2}+\frac{1-\tan\theta/2}{1+\tan\theta/2}$

$=\frac{(1+\tan\theta/2)^2+(1-\tan\theta/2)^2}{1-\tan^2\theta/2}=\frac{2(1+\tan^2\theta/2)}{1-\tan^2\theta/2}$

$=\frac{2}{\cos\theta}=\frac{2}{a/b}=\frac{2b}{a}$

36. (a) : We have, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

$\because \quad -\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2},\frac{-\pi}{2}\le\sin^{-1}y\le\frac{\pi}{2}$

and $\frac{-\pi}{2} \le \sin^{-1} z \le \frac{\pi}{2}$

∴ The above condition will satisfy if

$\sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \frac{\pi}{2} \Rightarrow x = y = z = 1$

37. (b) : We have, $6 \sin^{-1} (x^2 - 6x + 8.5) = \pi$

$\Rightarrow \quad \sin^{-1}(x^2 - 6x + 8.5) = \frac{\pi}{6}$

$\Rightarrow \quad x^2 - 6x + 8.5 = \sin\frac{\pi}{6} = \frac{1}{2}$

$\Rightarrow \quad x^2 - 6x + 8 = 0$

$\Rightarrow \quad (x - 4)(x - 2) = 0 \Rightarrow x = 4$ or $x = 2$

38. (b) : $\sin^{-1}(x^2 - 7x + 12) = 0$

$\Rightarrow \quad x^2 - 7x + 12 = \sin 0$

$\Rightarrow \quad x^2 - 7x + 12 = 0 \Rightarrow (x - 4)(x - 3) = 0 \Rightarrow x = 4, 3$

39. (a) : We have, $\tan^{-1}(x^2 - 4x + 4) = 0$

$\Rightarrow \quad x^2 - 4x + 4 = \tan 0 \Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow \quad (x - 2)^2 = 0 \Rightarrow x = 2$

40. (c) : $\tan^{-1}(\cot\theta) = 2\theta \Rightarrow \cot\theta = \tan 2\theta$

$\Rightarrow \quad \cot\theta = \cot\left(\frac{\pi}{2} - 2\theta\right) \Rightarrow \theta = \frac{\pi}{2} - 2\theta$

$\Rightarrow \quad 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$

41. (d) : Principal value branch of $\sec^{-1}x$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

42. (d) : $\cos\left(2\cos^{-1}\left(\frac{2}{5}\right)\right) = \cos 2x$, where $x = \cos^{-1}\frac{2}{5}$

$= 2\cos^2 x - 1 = 2\left(\frac{2}{5}\right)^2 - 1 \qquad \left(\because \cos x = \frac{2}{5}\right)$

$= \frac{2 \times 4}{25} - 1 = \frac{8 - 25}{25} = -\frac{17}{25}$

43. (d) : $\sin\left(\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1$

44. (a) : Let $\cot^{-1}\left(-\sqrt{3}\right) = \theta \Rightarrow \cot\theta = -\sqrt{3} = -\cot\frac{\pi}{6}$

$= \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6} \Rightarrow \theta = \frac{5\pi}{6} \in (0, \pi)$

∴ Principal value of $\cot^{-1}\left(-\sqrt{3}\right)$ is $\frac{5\pi}{6}$.

45. (a) : $\tan^{-1}(1) + \tan^{-1}(0) + \tan^{-1}(-1)$

$= \frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi$

46. (c) : Since, $\tan^{-1} x$ and $\cot^{-1} x$ exists for all $x \in R$ and $\cos^{-1}(2 - x)$ exists, if $-1 \le 2 - x \le 1 \Rightarrow 1 \le x \le 3$

∴ $\tan^{-1} x - \cot^{-1} x = \cos^{-1}(2 - x)$ is possible only if $1 \le x \le 3$

Thus, the solution of given equation is $[1, 3]$.

47. (d) : We have, $\cos^{-1}\left[\cos\left(2\cot^{-1}(\sqrt{3})\right)\right]$

$= \cos^{-1}\left[\cos 2\left(\frac{\pi}{6}\right)\right]$

$= \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

48. (a) : $2(\sin^{-1}x)^2 - 5(\sin^{-1}x) + 2 = 0$

$\Rightarrow \quad \sin^{-1} x = \frac{5 \pm \sqrt{25 - 16}}{4}$

$\Rightarrow \sin^{-1} x = \frac{1}{2}, \sin^{-1} x = 2$

$\Rightarrow \quad x = \frac{\pi}{6}$ is only solution

$\left[\because \frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}, \sin^{-1} x = 2 \text{ is not possible}\right]$

49. (d) : In $[0, \pi]$,

$\cos^{-1}\left(\cos\left(\frac{9\pi}{4}\right)\right) = \cos^{-1}\left(\cos\left(2\pi + \frac{\pi}{4}\right)\right)$

$= \left(\cos^{-1}\left(\cos\frac{\pi}{4}\right)\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

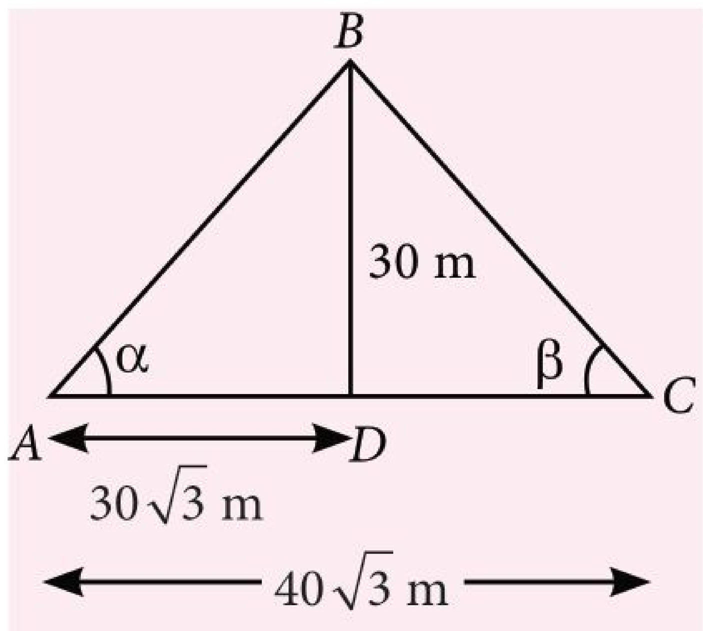
50. (d) : In $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$,

$\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(2\pi - \frac{\pi}{3}\right)\right)$

$= \sin^{-1}\left(\sin\left(\frac{-\pi}{3}\right)\right) = \frac{-\pi}{3}$

51. (b) : We have, $BD = 30$ m,

$AC = 40\sqrt{3}$ m, $AD = 30\sqrt{3}$ m,



Clearly, $BD \perp AC$

$\therefore$ In right ΔADB,

$AB^2 = AD^2 + BD^2 = (30\sqrt{3})^2 + (30)^2 = 3600$

$\Rightarrow$ $AB = 60$ m

Now, $\sin\alpha = \frac{DB}{AB} = \frac{30}{60} = \frac{1}{2}$

$\Rightarrow \alpha = \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \angle CAB = \alpha = \sin^{-1}\left(\frac{1}{2}\right)$

52. (c) : In right ΔADB, $\cos\alpha = \frac{AD}{AB} = \frac{30\sqrt{3}}{60} = \frac{\sqrt{3}}{2}$

$\Rightarrow \angle CAB = \alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

53. (d) : Clearly, $CD = AC - AD = 40\sqrt{3} - 30\sqrt{3} = 10\sqrt{3}$ m

$\therefore \tan\beta = \frac{BD}{CD} = \frac{30}{10\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\Rightarrow \angle BCA = \beta = \tan^{-1}(\sqrt{3})$

54. (c) : $\sin\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$; $\tan\beta = \sqrt{3} \Rightarrow \beta = \frac{\pi}{3}$

$\therefore \angle ABC = \pi - (\alpha + \beta) = \pi - \left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \frac{\pi}{2}$

55. (c) : Domain and range of $\cos^{-1}x$ are $[-1, 1]$ and $[0, \pi]$ respectively.

56. (b) : Let $\angle CAB = \alpha$, then $\angle DAB = 2\alpha$ and $\angle EAB = 3\alpha$

In right ΔCAB,

$\tan\alpha = \frac{BC}{AB} = \frac{10}{20} = \frac{1}{2} \Rightarrow \angle CAB = \alpha = \tan^{-1}\left(\frac{1}{2}\right)$

57. (c) : $\angle DAB = 2\alpha = 2\tan^{-1}(1/2)$

58. (d) : $\angle EAB = 3\alpha = 3\tan^{-1}(1/2)$

59. (b) : In right $\Delta CA'B$,

$\tan\angle CA'B = \frac{BC}{A'B} = \frac{10}{25} = \frac{2}{5} \Rightarrow \angle CA'B = \tan^{-1}\left(\frac{2}{5}\right)$

$\therefore$ Required difference $= \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{2}{5}\right)$

60. (c) : Domain and range of $\tan^{-1}x$ are R and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively.

61. (a) : Let $y = \sin^{-1}(\sin(\pi - \pi/3))$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\left[\because \sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

62. (c) : We have, $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$

Clearly, domain of $f(x)$ is given by $x = \pm 1$.

Thus, the range is $\{f(-1), f(1)\}$, *i.e.*, $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

63. (a) : Reason is correct, from which we can say $\cot^{-1}x + \cos^{-1}2x = -\pi$ is not possible. Hence, both the Statements are correct and Statement-II is the correct explanation of Statement-I.

64. (d) : Let $\theta = \cot^{-1}(2x - x^2)$, where $\theta \in (0, \pi)$

$\Rightarrow$ $\cot\theta = 2x - x^2$, where $\theta \in (0, \pi)$

$= 1 - (1 - 2x + x^2)$, where $\theta \in (0, \pi)$

$= 1 - (1 - x)^2$, where $\theta \in (0, \pi)$

$\Rightarrow$ $\cot\theta \le 1$, where $\theta \in (0, \pi)$

$\Rightarrow \frac{\pi}{4} \le \theta < \pi \Rightarrow$ Range of $f(x)$ is $\left[\frac{\pi}{4}, \pi\right)$

65. (d) : $\sin^{-1}(x)$ is defined if $-1 \le x \le 1$.

But $1 + x^2 \ge 1$ and equality holds when $x = 0$

$\therefore$ Domain of $f(x)$ is $\{0\}$.

66. (c) : Let $2\tan^{-1}(0.75) = \theta \Rightarrow 0.75 = \tan\left(\frac{\theta}{2}\right)$

$\therefore$ $\sin(2\tan^{-1}(0.75))$

$= \sin\theta = \frac{2\tan\theta/2}{1 + \tan^2\theta/2} = \frac{2 \times 0.75}{1 + (0.75)^2} = \frac{1.50}{1.5625} = 0.96$

67. (d) : Clearly, Statement-II is correct statement.

Now, consider $\cos\left(\frac{\pi}{3} - \cos^{-1}\left(\frac{1}{2}\right)\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right)$

$= \cos 0 = 1$

$\therefore$ Statement-I is wrong statement.

68. (b) : We have, $4\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 4 \cdot \frac{\pi}{6} = \frac{2\pi}{3}$.

$\therefore$ Statement-I is correct statement.

Thus, Statement-I and Statement-II are correct statements but Statement-II is not the correct explanation of Statement-I.

69. (c) : Clearly, $f(x)$ will be defined, if $0 \le x - 1 \le 1$

$\Rightarrow$ $1 \le x \le 2$

$\therefore$ Domain of $f(x)$ is $[1, 2]$.

$\therefore$ Statement-I is true statement.

But, Statement-II is false statement.

70. (a) : Let $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$. Then Dom $(f) = [-1, 1] \cap [-1, 1] \cap R = [-1, 1]$.

$\therefore$ Both Statements are correct and Statement-II is the correct explanation of Statement-I.

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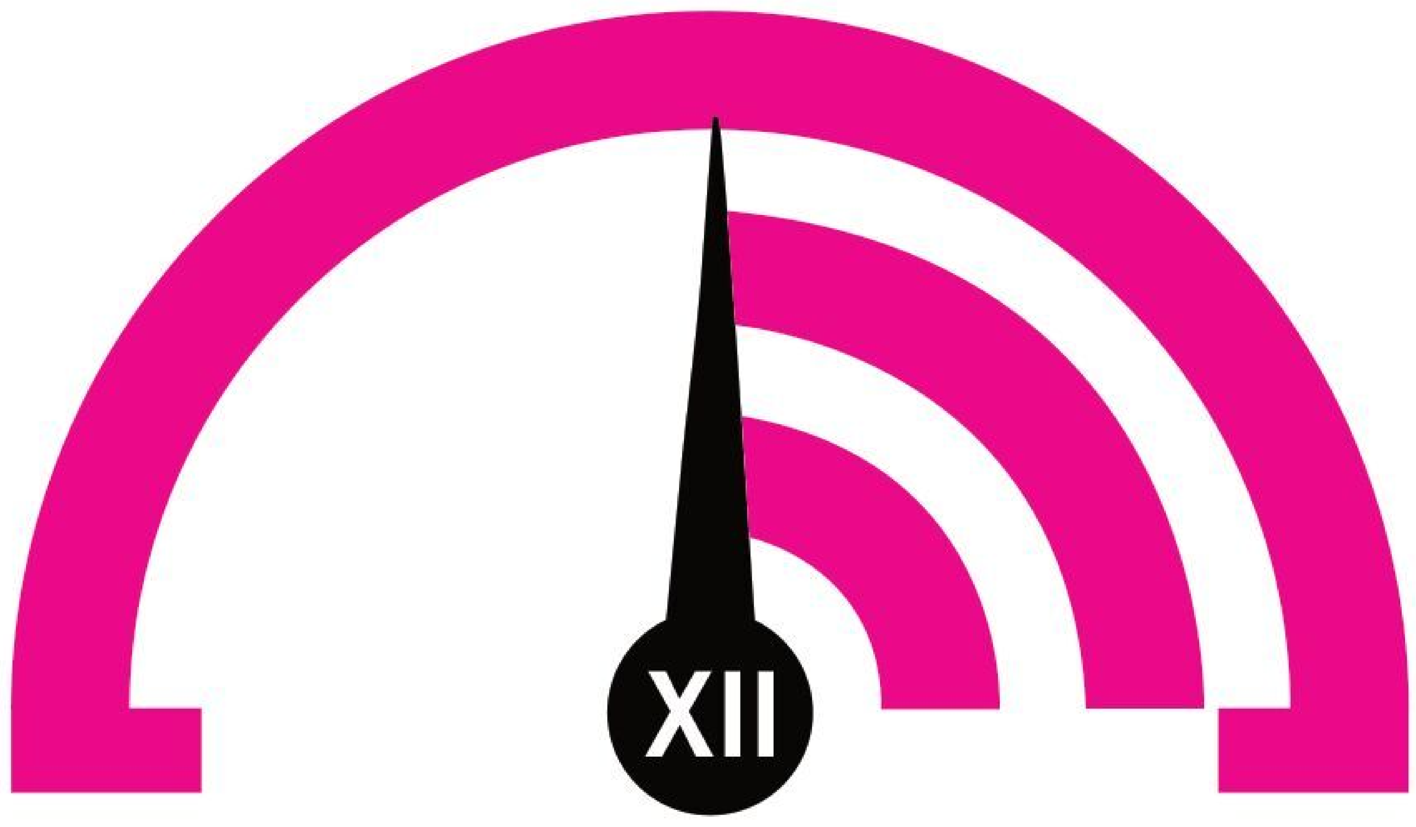


MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 3 : Matrices and Determinants

Time Taken : 60 Min.

Only One Option Correct Type

1. Let $\Delta(x)=\begin{vmatrix}(x-2) & (x-1)^2 & x^3\\ (x-1) & x^2 & (x+1)^3\\ x & (x+1)^2 & (x+2)^3\end{vmatrix}$, then the coefficient of x in $\Delta(x)$ is

(a) -3 (b) -2
(c) -1 (d) 0

2. Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$.

Then $\mathrm{tr}(A) - \mathrm{tr}(B)$ has the value equal to

(a) 0 (b) 1
(c) 2 (d) 3

3. Let $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}$. If $10A^{10}$ + $\mathrm{adj}(A^{10}) = B$, then $b_1 + b_2 + b_3 + b_4$ is equal to

(a) 91 (b) 92
(c) 111 (d) 112

4. If a, b, c are in G.P. and $(x - \alpha)$ is a factor of $ax^2 + 2bx + c = 0$, then value of the determinant $\begin{vmatrix}a & b & a\alpha+b\\ b & c & b\alpha+c\\ a\alpha+b & b\alpha+c & 0\end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

5. If $A^5 = O$ such that $A^n \neq I$ for $1 \leq n \leq 4$, then $(I - A)^{-1}$ is equal to

(a) A^4 (b) A^3
(c) $I + A$ (d) None of these

6. If $D = \mathrm{diag}\,(d_1, d_2, d_3, ..., d_n)$, where $d_i \geq 0\ \forall\ i$, then D^{-1} equals

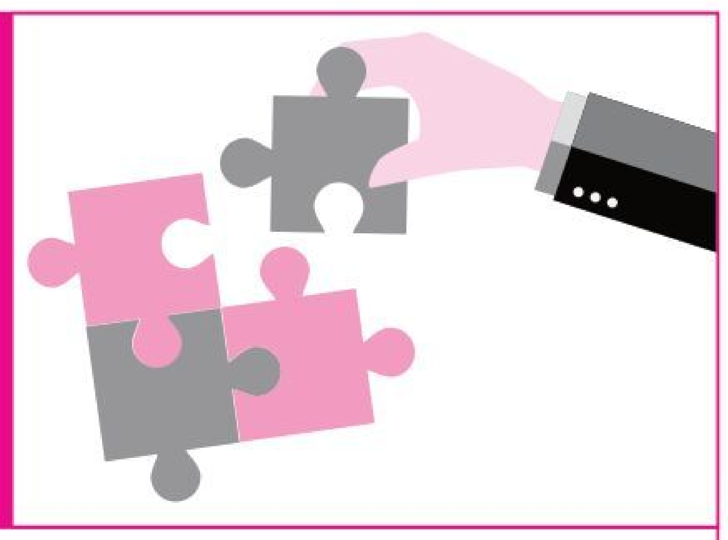
(a) D
(b) $\mathrm{diag}\,(d_1^n, d_2^n, ..., d_n^n)$
(c) I_n
(d) $\mathrm{diag}\,(d_1^{-1}, d_2^{-1}, d_3^{-1}, ..., d_n^{-1})$

One or More than One Option(s) Correct Type

7. If $f(x)=\begin{vmatrix}a & -1 & 0\\ ax & a & -1\\ ax^2 & ax & a\end{vmatrix}$, then $f(2x) - f(x)$ is divisible by

(a) x (b) a
(c) $2a + 3x$ (d) x^2

PUZZLE CORNER



ANSWER - AUGUST 2021

1− 6	2− 4	2− 5	3+ 2	1	2− 3
5	6	3	2÷ 4	2	1
2− 4	4+ 1	2	11+ 5	3	16+ 6
2	10× 5	1	3	6	4
9× 3	2	10+ 6	5− 1	1− 4	5
1	3	4	6	3− 5	2

8. If $A + B + C = \pi$, $e^{i\theta} = \cos\theta + i\sin\theta$ and

$$z = \begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}, \text{ then}$$

(a) Re $(z) = 4$ (b) Im $(z) = 0$
(c) Re $(z) = -4$ (d) Im $(z) = -1$

9. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when $n =$

(a) 57 (b) 55 (c) 58 (d) 56

10. Let $a, \lambda, \mu \in R$. Consider the system of linear equations $ax + 2y = \lambda$, $3x - 2y = \mu$. Which of the following statement(s) is(are) correct?

(a) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ.
(b) If $a \neq -3$, then the system has a unique solution for all values of λ and μ.
(c) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$.
(d) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$.

11. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ be a matrix. If $A^{10} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

(a) number of divisors of a is 11
(b) a is an odd integer
(c) $(a + b + d)$ is an even integer
(d) $a + d$ is a multiple of 13

12. The value of θ for which the system of linear equations in x, y, z given as

$(\sin 3\theta)\, x - y + z = 0$
$(\cos 2\theta)x + 4y + 3z = 0$
$2x + 7y + 7z = 0$

has a non-trivial solution, is/are

(a) $\frac{n\pi}{2}$ (b) $\frac{2n\pi}{3}$
(c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) None of these

13. If A is symmetric and B is skew symmetric matrix, then which of the following is/are not true?

(a) ABA^T is symmetric matrix
(b) $AB^T + BA^T$ is symmetric matrix
(c) $(A + B)(A - B)$ is skew symmetric
(d) $(A + I)(B - I)$ is skew symmetric

Comprehension Type

Paragraph for Q. No. 14 and 15

If $x > m, y > n, z > r$ $(x, y, z > 0)$ such that $\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0.$

14. The value of $\frac{x}{x-m}+\frac{y}{y-n}+\frac{z}{z-r}$ is

(a) 2 (b) –4

(c) 0 (d) –1

15. The greatest value of $\frac{xyz}{(x-m)(y-n)(z-r)}$ is

(a) 27 (b) 8/27

(c) 64/27 (d) None of these

Matrix Match Type

16. Match the following:

Column-I		Column-II	
P.	If ω is a non real cube root of unity, then a root of the following equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$ is	1.	0
Q.	If $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{bmatrix}$, then rank of A is	2.	–2
R.	Suppose that system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has non trivial solution. Then $a^2 + b^2 + c^2 + 2abc =$	3.	1
S.	$\begin{vmatrix} a+1 & a+2 & a+4 \\ a+3 & a+5 & a+8 \\ a+7 & a+10 & a+14 \end{vmatrix} =$	4.	4

	P	Q	R	S
(a)	1	3	3	2
(b)	2	4	3	1
(c)	4	2	3	1
(d)	4	1	2	3

Numerical Answer Type

17. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then the value of $64(\sin^2 A + \sin^2 B + \sin^2 C)$ must be ________.

18. Let S be the set which contains all possible values of l, m, n, p, q, r for which $A = \begin{bmatrix} l^2-3 & p & 0 \\ 0 & m^2-8 & q \\ r & 0 & n^2-15 \end{bmatrix}$ be a non singular idempotent matrix. Then the sum of all the elements of the set S is ________.

19. If A, B and C are $n \times n$ matrices and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then the value of $[\det(A^2BC^{-1})]$ (where $[\cdot]$ represents greatest integer function) is _________.

20. If $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $M^2 - \lambda M - I_2 = O$, λ must be _________.

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Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......

No. of questions correct ......

Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.